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Error Bounds for Spline and L-Spline Interpolation

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1. INTRODUCTION

One of our basic aims here is to obtain improved error bounds for spline and L-spline interpolation at knots, and to obtain certain stability (or perturbation) results for such forms of interpolation. To give a concrete example to illustrate our aim, consider for simplicity the interpolation of a given function f defined on $[a, b]$ by a smooth cubic spline s over a uniform partition Δ_u of $[a, b]$. Normally, if $f \in C^1[a, b]$, then its unique cubic spline interpolant s is defined by

$$\begin{aligned} s(a + ih) &= f(a + ih), \qquad 0 \leqslant i \leqslant N, \qquad h = (b - a)/N, \\ Ds(a) &= Df(a), \qquad Ds(b) = Df(b), \qquad D \equiv d/dx. \end{aligned} \tag{1.1}$$

If, on the other hand, f is only continuous on $[a, b]$, the second part of (1.1) must be modified; for example, one may assume that $Ds(a) = Ds(b) = 0$. However, from a computational point of view, it would be advantageous to have a *single* definition for $Ds(a)$ and for $Ds(b)$ which does not depend explicitly on the continuity class of f, and for which optimal interpolation errors are obtained. For example, suppose that we define the cubic spline interpolant s of f by means of

$$\begin{aligned} &s(a + ih) = f(a + ih), \qquad 0 \leqslant i \leqslant N, \\ &Ds(a) = (1/6h)\{-11f(a) + 18f(a + h) - 9f(a + 2h) + 2f(a + 3h)\}, \\ &Ds(b) = (1/6h)\{11f(b) - 18f(b - h) + 9f(b - 2h) - 2f(b - 3h)\}. \end{aligned} \tag{1.2}$$

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Because the definitions of $Ds(a)$ and $Ds(b)$ come from the derivative of a cubic Lagrange polynomial interpolation of f in particular knots, it will be shown in Section 7 that there exists a constant K, independent of f and h, such that if $f \in C^k[a, b]$, $0 \leqslant k < 4$, then

$$Kh^{k-j}\omega(D^k f, h) \geqslant \begin{cases} \| D^j(f - s)\|_{L_\infty[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_\infty[a,b]}, & \text{if} \quad k < j \leqslant 3. \end{cases} \tag{1.3}$$

Moreover, we will show that error bounds similar to those of (1.3) are valid if s interpolates values close to those of f (cf. Theorem 7.6); we shall call such results *stability results.*

Another basic aim here is to obtain new interpolation error bounds for general L-splines (cf. Theorem 3.5), to obtain new interpolation error bounds in the *uniform* norm for splines defined on uniform partitions of $[a, b]$ (cf. Theorem 7.4), and to obtain analogous interpolation error bounds in the uniform norm for Hermite L-splines (cf. Theorem 6.1), along with stability results related to such error bounds. In Section 8, extensions of these results for polynomial splines to more general boundary conditions are treated. In so doing, we shall fill some gaps in the existing literature for such interpolation error bounds. A survey of the relevant literature concerning polynomial-spline approximations is also contained in Section 7.

2. Notation

For $-\infty < a < b < +\infty$ and for a positive integer N, let

$$\Delta : a = x_0 < x_1 < \cdots < x_N = b \tag{2.1}$$

denote a *partition* of $[a, b]$ with *knots* x_i. The collection of all such partitions Δ of $[a, b]$ is denoted by $\mathscr{P}(a, b)$. We further define $\pi \equiv \max_{0 \leqslant i \leqslant N-1} (x_{i+1} - x_i)$ and $\underline{\pi} \equiv \min_{0 \leqslant i \leqslant N-1} (x_{i+1} - x_i)$ for each partition Δ of the form (2.1). For any real number σ with $\sigma \geqslant 1$, $\mathscr{P}_\sigma(a, b)$ then denotes the subset of all partitions Δ in $\mathscr{P}(a, b)$ for which

$$\pi / \underline{\pi} \leqslant \sigma. \tag{2.2}$$

In particular, $\mathscr{P}_1(a, b)$ is the collection of all *uniform* partitions of $[a, b]$, and its elements are denoted by Δ_u.

Since we shall make extensive use of L-splines, we now briefly describe them. Given the differential operator L of order m,

$$Lu(x) \equiv \sum_{j=0}^{m} c_j(x)\, D^j u(x), \qquad m \geqslant 1, \qquad D^j \equiv (d/dx)^j, \tag{2.3}$$

where $c_j \in C^j[a, b]$, $0 \leqslant j \leqslant m$, with $c_m(x) \geqslant \delta > 0$ for all $x \in [a, b]$, and given the partition Δ of (2.1), for $N > 1$ let $z = (z_1, z_2, ..., z_{N-1})$, the *incidence vector*, be an $(N-1)$-tuple of positive integers with $1 \leqslant z_i \leqslant m$, $1 \leqslant i \leqslant N-1$. Then, $\mathrm{Sp}(L, \Delta, z)$, the *L-spline space*, is the collection of all real-valued functions w defined on $[a, b]$ such that (cf. Ahlberg, Nilson, and Walsh [3, Chapter 6] and Schultz and Varga [32])

$$\begin{aligned} L^*Lw(x) &= 0, \qquad x \in (a, b) - \{x_i\}_{i=1}^{N-1}, \\ D^k w(x_i-) &= D^k w(x_i+) \quad \text{for } 0 \leqslant k \leqslant 2m-1-z_i, \quad 0 < i < N, \end{aligned} \tag{2.4}$$

where L^* is the formal adjoint of L. From (2.4), we see that

$$\mathrm{Sp}(L, \Delta, z) \subset C^{2m-\mu-1}[a, b]$$

where $\mu \equiv \max_{1 \leqslant i \leqslant N-1} z_i$.

In the special case $L = D^m$, the elements of $\mathrm{Sp}(D^m, \Delta, z)$ are, from (2.4), polynomials of degree $(2m-1)$ on each subinterval of Δ, and as such are called *polynomial splines*. More specially, when $L = D^m$ and

$$z_i \equiv m, 0 < i < N,$$

the associated L-spline space is called the *Hermite space*, and is denoted by $H^{(m)}(\Delta)$. From (2.4), $H^{(m)}(\Delta) \subset C^{m-1}[a, b]$. Similarly, when $L = D^m$ and $z_i \equiv 1, 0 < i < N$, the associated L-spline space is called the *spline space*, and is denoted by $\mathrm{Sp}^{(m)}(\Delta)$. Again from (2.4), $\mathrm{Sp}^{(m)}(\Delta) \subset C^{2m-2}[a, b]$.

While our discussion makes explicit use of L-splines, it will be clear that most of the techniques used here in obtaining our L-spline interpolation error bounds can be modified to apply to the more general definitions of splines appearing in Jerome and Pierce [18], Jerome and Schumaker [19], Lucas [21], and Varga [40]. For the definitions and properties of these more general splines, we refer to the previously cited references.

As measures of the smoothness of the functions we interpolate, we recall first the Sobolev norm

$$\| f \|_{W_q^m[a,b]} \equiv \sum_{j=0}^{m} \| D^j f \|_{L_q[a,b]}, \tag{2.5}$$

where the Sobolev space $W_q^m[a, b]$, for m a positive integer and $1 \leqslant q \leqslant \infty$, is the collection of all real-valued functions $f(x)$ defined on $[a, b]$ with $D^{m-1}f$ absolutely continuous on $[a, b]$, and with $D^m f \in L_q[a, b]$. Next, if given $\Delta \in \mathscr{P}(a, b)$ and if $D^j g \in L_r[x_i, x_{i+1}]$ for each subinterval $[x_i, x_{i+1}]$, we use the following modified definition,

$$\| D^j g \|_{L_r[a,b]} = \left\{ \sum_{i=0}^{N-1} \| D^j g \|_{L_r[x_i, x_{i+1}]}^r \right\}^{1/r}, \tag{2.6}$$

of the L_r-norm of $D^j g$ over $[a, b]$, even if $D^j g \notin L_r[a, b]$. Next, if f is any bounded function defined on $[a, b]$, then

$$\omega(f, h) \equiv \sup\{|f(x + t) - f(x)| : x, x + t \text{ in } [a, b] \text{ and } |t| \leqslant h\} \tag{2.7}$$

denotes the usual *modulus of continuity* of f.

Finally, we shall throughout denote any generic constant which is independent of the functions considered and is independent of the maximum mesh spacing π, by the general symbol K. However, these constants in general *do* depend in particular upon m, $(b - a)$, and the order of various derivatives considered, as well as upon the constant σ if $\Delta \in \mathscr{P}_\sigma(a, b)$.

3. Basic Comparison Functions

The idea behind the proofs to follow is an elementary one, based on the triangle inequality. From known interpolation errors for smooth functions g (e.g., Lemma 3.1), error bounds for less smooth functions f are determined as follows. A smooth piecewise polynomial interpolation g of f is constructed, and bounds for $f - g$ are determined (Lemma 3.2). A spline interpolate, s, of f is then defined, which is also a spline interpolant of this smooth g. Then, bounds for $f - s$ will follow from known bounds for $f - g$ and $g - s$.

To begin, given any $g \in C^{m-1}[a, b]$, it is known (cf. Schultz and Varga [32]) that there exists a unique $s \in \mathrm{Sp}(L, \Delta, z)$ such that

$$\begin{aligned} &D^j(g - s)(x_i) = 0, \qquad 0 \leqslant j \leqslant z_i - 1, \quad \text{if} \quad 0 < i < N, \\ &D^j(g - s)(a) = D^j(g - s)(b) = 0, \qquad 0 \leqslant j \leqslant m - 1, \end{aligned} \tag{3.1}$$

and s is said to interpolate g under Hermite boundary conditions, corresponding to the fact that s interpolates g and all its derivatives at the endpoints of [a, b]. Because such boundary interpolation insures the *second integral relation* (cf. Ahlberg, Nilson, and Walsh [3, p. 205]) the following error bounds are typical (cf. Hedstrom and Varga [16, Theorem 3.6]), and follow from results of Jerome and Varga [20], Schultz and Varga [32], and Perrin [25].

Lemma 3.1. *Given $g \in W_2^{2m}[a, b]$ and given $\Delta \in \mathscr{P}_\sigma(a, b)$, let s be the unique element in* $\mathrm{Sp}(L, \Delta, z)$ *which interpolates g in the sense of* (3.1). *Then, for* $2 \leqslant q \leqslant \infty$,

$$\| D^j(g - s)\|_{L_q[a,b]} \leqslant K\pi^{2m-j-(1/2)+(1/q)} \| g \|_{W_2^{2m}[a,b]}, \qquad 0 \leqslant j \leqslant 2m - 1. \tag{3.2}$$

For polynomial splines, i.e., $L = D^m$, $\| g \|_{W_2^{2m}[a,b]}$ *can be replaced by*

$$\| D^{2m} g \|_{L_2[a,b]}$$

in (3.2).

We remark that such error bounds (3.2) are in fact valid for more general boundary interpolation (cf. Schultz and Varga [32, Theorem 8]), as well as for the more general Besov spaces (cf. Hedstrom and Varga [16]). Generalizations to other boundary interpolation will be given in Section 8, but generalizations to Besov spaces, which follow rather easily from the results of [16], will not be considered here. It is interesting to note that the proof of Lemma 3.1 requires only the tools of advanced calculus.

We now construct a polynomial spline $g \in H^{(2m+1)}(\Delta)$ which is close to a given (not necessarily smooth) function f defined on $[a, b]$.

LEMMA 3.2. *Given* $f \in C^k[a, b]$ *with* $0 \leqslant k < 2m$ *and given* $\Delta \in \mathscr{P}_\sigma(a, b)$, *let g be the unique element in* $H^{(2m+1)}(\Delta)$ *such that*

$$\begin{aligned} D^j(f-g)(x_i) &= 0 \qquad 0 \leqslant j \leqslant k, \quad 0 \leqslant i \leqslant N, \\ D^j g(x_i) &= 0, \qquad k < j \leqslant 2m, \quad 0 \leqslant i \leqslant N. \end{aligned} \tag{3.3}$$

Then, $g \in C^{2m}[a, b]$ *and*

$$K\pi^{k-j}\omega(D^k f, \pi) \geqslant \begin{cases} \| D^j(f-g) \|_{L_\infty[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j g \|_{L_\infty[a,b]}, & k < j \leqslant 2m. \end{cases} \tag{3.4}$$

Proof. Given $4m + 2$ arbitrary real numbers α_i, β_i, $0 \leqslant i \leqslant 2m$, and any finite interval $[c, d]$ with $c < d$, it is well known that there exists a unique polynomial $p(x)$ of degree at most $(4m + 1)$ such that $D^j p(c) = \alpha_j$, $D^j p(d) = \beta_j$, $0 \leqslant j \leqslant 2m$. From this, it follows that a unique polynomial spline $g \in H^{(2m+1)}(\Delta)$ exists, satisfying (3.3). It thus remains to prove (3.4).

First we establish the special case $k = 0$ of (3.4). Let $\phi_{0,l}(x)$ and $\phi_{1,l}(x)$ be the unique polynomials of degree $(4m + 1)$ (cf. the proof of Theorem 7.4) such that

$$D^j\phi_{i,l}(0) = \delta_{j,l}\delta_{i,0}, \quad D^j\phi_{i,l}(1) = \delta_{j,l}\delta_{i,1}, \quad 0 \leqslant j \leqslant 2m, \quad i = 0, 1, \tag{3.5}$$

where $\delta_{i,j}$ denotes the Kronecker δ function. It is readily verified that

$$\phi_{0,0}(x) \geqslant 0, \quad \phi_{1,0}(x) \geqslant 0, \quad \text{and} \quad \phi_{0,0}(x) + \phi_{1,0}(x) = 1 \text{ for } x \in [0, 1], \tag{3.6}$$

so that $\phi_{0,0}$ and $\phi_{1,0}$ form a *partition of unity*. If $[x_i, x_{i+1}]$ is any subinterval of $[a, b]$ determined from $\Delta \in \mathscr{P}_\sigma(a, b)$, it follows from (3.3) that g can be expressed as

$$g(x) = f(x_i)\, \phi_{0,0}\left(\frac{x - x_i}{h_i}\right) + f(x_{i+1})\, \phi_{1,0}\left(\frac{x - x_i}{h_i}\right), \qquad x \in [x_i, x_{i+1}], \quad (3.7)$$

where $h_i \equiv x_{i+1} - x_i$. Since $\phi_{0,0} + \phi_{1,0} \equiv 1$, we can express $f(x)$ as

$$f(x) = f(x)\, \phi_{0,0}\left(\frac{x - x_i}{h_i}\right) + f(x)\, \phi_{1,0}\left(\frac{x - x_i}{h_i}\right), \qquad x \in [x_i, x_{i+1}].$$

Thus,

$$\begin{aligned} |(f-g)(x)| &\leqslant |f(x) - f(x_i)|\, \phi_{0,0}\left(\frac{x - x_i}{h_i}\right) + |f(x) - f(x_{i+1})|\, \phi_{1,0}\left(\frac{x - x_i}{h_i}\right) \\ &\leqslant \omega(f, h_i) \qquad \text{for} \quad x \in [x_i, x_{i+1}], \end{aligned}$$

using the definition of the modulus of continuity ω in (2.7). But as

$$\omega(f, h_i) \leqslant \omega(f, \pi),$$

then

$$\| f - g \|_{L_\infty[a,b]} \leqslant \omega(f, \pi),$$

the desired first inequality of (3.4) for the case $k = 0$. Next, again using the fact that $\phi_{0,0} + \phi_{1,0} \equiv 1$, it follows from (3.7) that

$$D^j g(x) = h_i^{-j}(f(x_i) - f(x_{i+1}))\, D^j \phi_{0,0}\left(\frac{x - x_i}{h_i}\right), \qquad x \in [x_i, x_{i+1}],$$

and hence,

$$\| D^j g \|_{L_\infty[x_i, x_{i+1}]} \leqslant K h_i^{-j} \omega(f, \pi), \qquad 1 \leqslant j \leqslant 2m.$$

But, as $h_i^{-j} \leqslant (\underline{\pi})^{-j} \leqslant (\pi/\sigma)^{-j}$ since $\Delta \in \mathscr{P}_\sigma(a, b)$, [cf. (2.2)], we thus have

$$\| D^j g \|_{L_\infty[a,b]} \leqslant K \pi^{-j} \omega(f, \pi), \qquad 1 \leqslant j \leqslant 2m,$$

the desired second inequality of (3.4) for the case $k = 0$.

Suppose now that $k \geqslant 1$. From Taylor's formula, we can write for $x \in [x_i, x_{i+1}]$ that

$$f(x) = \sum_{j=0}^{k-1} \frac{D^j f(x_i)}{j!} (x - x_i)^j + \frac{h_i^k}{(k-1)!} \int_0^1 \left(\frac{x - x_i}{h_i} - t\right)_+^{k-1} D^k f(x_i + h_i t)\, dt,$$

where as usual

$$(y - t)_+^{k-1} \equiv \begin{cases} (y - t)^{k-1} & \text{for } \; y \geqslant t, \\ 0 & \text{for } \; y < t. \end{cases}$$

Because

$$\frac{h_i^k}{(k-1)!}\int_0^1 \left(\frac{x-x_i}{h_i}-t\right)_+^{k-1} dt = \frac{1}{k!}(x-x_i)^k,$$

we can also express $f(x)$ as

$$f(x) = \sum_{j=0}^{k-1} \frac{D^j f(x_i)}{j!}(x-x_i)^j + \frac{\alpha}{k!}(x-x_i)^k$$
$$+ \frac{h_i^k}{(k-1)!}\int_0^1 \left(\frac{x-x_i}{h_i}-t\right)_+^{k-1} \{D^k f(x_i+h_i t)-\alpha\}\, dt, \quad (3.8)$$

for any scalar α. Next, for each fixed $t_0 \in (0,1)$, let $Q(y;t_0)$ be the polynomial interpolation, as a function of y, of $(y-t_0)_+^{k-1}$ such that $Q(y,t_0)$ is a polynomial of degree $(4m+1)$ in y with

$$D_y^j Q(y_l;t_0) = \begin{cases} D_y^j(y_l-t_0)_+^{k-1}, & 0 \leqslant j < k \\ 0, & k \leqslant j \leqslant 2m \end{cases}, \qquad l = 0 \text{ or } 1,$$

where $y_0 = 0$ and $y_1 = 1$, and D_y denotes differentiation with respect to the first variable, y. Because this interpolation of (3.3) is linear and in fact exact for polynomials of degree at most k, we have, in the manner of the Peano kernel theorem (cf. Sard [26, p. 14]) that the function g of (3.3) can be expressed on $[x_i, x_{i+1}]$ for any scalar α as

$$g(x) = \sum_{j=0}^{k-1} \frac{D^j f(x_i)}{j!}(x-x_i)^j + \frac{\alpha}{k!}(x-x_i)^k$$
$$+ \frac{h_i^k}{(k-1)!}\int_0^1 Q\left(\frac{x-x_i}{h_i};t\right)\{D^k f(x_i+h_i t)-\alpha\}\, dt$$
$$+ h_i^k\left[(D^k f(x_i)-\alpha)\,\phi_{0,k}\left(\frac{x-x_i}{h_i}\right)\right.$$
$$\left.+ (D^k f(x_{i+1})-\alpha)\,\phi_{1,k}\left(\frac{x-x_i}{h_i}\right)\right]. \quad (3.9)$$

Subtracting (3.9) from (3.8) gives us, after differentiation,

$$D^j(f-g)(x) = \frac{h_i^{k-j}}{(k-1)!}\int_0^1 \left\{D_y^j\left[\left(\frac{x-x_i}{h_i}-t\right)_+^{k-1} - Q\left(\frac{x-x_i}{h_i};t\right)\right]\right\}$$
$$\times \{D^k f(x_i+h_i t)-\alpha\}\, dt$$
$$- h_i^{k-j}\left[(D^k f(x_i)-\alpha)\, D^j\phi_{0,k}\left(\frac{x-x_i}{h_i}\right)\right.$$
$$\left.+ (D^k f(x_{i+1})-\alpha)\, D^j\phi_{1,k}\left(\frac{x-x_i}{h_i}\right)\right], \quad (3.10)$$

for $0 \leqslant j \leqslant k-1$. Because $D^j\phi_{i,k}(y)$ is bounded in $[0, 1]$ and

$$D_y^j\{(y-t)_+^{k-1} - Q(y; t)\}$$

is *uniformly bounded* in $[0, 1] \times [0, 1]$ for all $0 \leqslant j \leqslant k-1$, we see from (3.10) that for $x \in [x_i, x_{i+1}]$,

$$\begin{aligned} | D^j(f-g)(x)| \leqslant K h_i^{k-j} \Big\{ \int_0^1 | D^k f(x_i + h_i t) - \alpha | \, dt \\ + | D^k f(x_i) - \alpha | + | D^k f(x_{i+1}) - \alpha | \Big\}. \end{aligned} \tag{3.11}$$

Hence, upon choosing $\alpha = D^k f(x)$, we have

$$\| D^j(f-g)\|_{L_\infty[x_i, x_{i+1}]} \leqslant K h_i^{k-j} \omega(D^k f, \pi), \qquad 0 \leqslant j \leqslant k-1,$$

from which the first inequality of (3.4) follows for $0 \leqslant j \leqslant k-1$.

For the case $j = k \geqslant 1$, we have using (3.9) that

$$\begin{aligned} D^k(f-g)(x) = (D^k f(x) - \alpha) - \frac{1}{(k-1)!} \int_0^1 D_y^k Q\left(\frac{x-x_i}{h}; t\right) \\ \times \{D^k f(x_i + h_i t) - \alpha\} \, dt \\ - [D^k f(x_i) - \alpha] \, D^k \phi_{0,k}\left(\frac{x-x_i}{h_i}\right) \\ - [D^k f(x_{i+1}) - \alpha] \, D^k \phi_{1,k}\left(\frac{x-x_i}{h_i}\right). \end{aligned}$$

Again, choosing $\alpha = D^k f(x)$, we see that each of the above terms can be bounded above in terms of $\omega(D^k f, \pi)$, so that

$$\| D^k(f-g)\|_{L_\infty[x_i, x_{i+1}]} \leqslant K \omega(D^k f, \pi), \qquad 0 \leqslant i \leqslant N-1,$$

from which the special case $j = k$ of (3.4) follows.

For the case $1 \leqslant k < j \leqslant 2m$, it follows from (3.9) that

$$\begin{aligned} D^j g(x) = h_i^{k-j} \Big\{ \int_0^1 D_y^j Q\left(\frac{x-x_i}{h}; t\right) \{D^k f(x_i + h_i t) - \alpha\} \, dt/(k-1)! \\ + [D^k f(x_i) - \alpha] \, D^j \phi_{0,k}\left(\frac{x-x_i}{h_i}\right) \\ + [D^k f(x_{i+1}) - \alpha] \, D^j \phi_{1,k}\left(\frac{x-x_i}{h_i}\right) \Big\}. \end{aligned} \tag{3.12}$$

Because $D^j\phi_{i,k}(y)$ is bounded in [0, 1] and $D_y{}^jQ(y; t)$ is uniformly bounded in $[0, 1] \times [0, 1]$ for all $0 \leqslant j \leqslant 2m$, the choice $\alpha = D^kf(x)$ then gives

$$\| D^jg \|_{L_\infty[x_i,x_{i+1}]} \leqslant Kh_i^{k-j}\omega(D^kf, \pi), \qquad 0 \leqslant i \leqslant N-1.$$

Again, since $\Delta \in \mathscr{P}_\sigma(a, b)$, the above inequality can be extended over $[a, b]$ in the manner of (2.6), to give

$$\| D^jg \|_{L_\infty[a,b]} \leqslant K\pi^{k-j}\omega(D^kf, \pi), \qquad k < j \leqslant 2m,$$

the desired second inequality of (3.4). Q.E.D.

Since by definition $W_r^{k+1}[a, b] \subset C^k[a, b]$ for any $1 \leqslant r \leqslant \infty$, the results of Lemma 3.2 directly apply to the elements of $W_r^{k+1}[a, b]$. In this case, the modulus of continuity $\omega(D^kf, \pi)$ can be suitably bounded above as follows. For any $f \in W_r^{k+1}[a, b]$ and any $x, y \in [a, b]$ with $0 \leqslant |y - x| \leqslant \pi$, we see from Hölder's inequality that

$$| D^k(f(y) - f(x))| = \left| \int_x^y D^{k+1}f(t)\, dt \right| \leqslant | y - x |^{1-(1/r)} \| D^{k+1}f \|_{L_r[a,b]},$$

so that

$$\omega(D^kf, \pi) \leqslant \pi^{1-(1/r)} \| D^{k+1}f \|_{L_r[a,b]}. \tag{3.13}$$

Substituting the above inequality in (3.4) then gives the error bounds of (3.14) in the uniform norm, $q = \infty$. In a similar way beginning with (3.11) and using Jensen's inequality, as in Birkhoff, Schultz, and Varga [7, Theorem 2], we also obtain error bounds in the L_q norm. We state this as

COROLLARY 3.3. *With the hypotheses of Lemma* 3.2, *if* $f \in W_r^{k+1}[a, b]$ *with* $0 \leqslant k < 2m$ *and* $1 \leqslant r \leqslant \infty$, *then for* $r \leqslant q \leqslant \infty$,

$$K\pi^{k+1-j-(1/r)+(1/q)} \| D^{k+1}f \|_{L_r[a,b]} \geqslant \begin{cases} \| D^j(f - g) \|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^jg \|_{L_q[a,b]}, & k < j \leqslant 2m. \end{cases} \tag{3.14}$$

Actually, a more general form of Corollary 3.3 can be established, which is closely related to the recent interpolation results of Golomb [15] for functions f with D^kf *piecewise continuous* on $[a, b]$. We give this in Corollary 3.4 below. See also Swartz [39, Corollary 4.3] for the corresponding generalization of Lemma 3.2.

COROLLARY 3.4. *Given* $f(x)$ *defined on* $[a, b]$ *such that* $D^{k-1}f$ (*for* $k \geqslant 1$) *is absolutely continuous and* D^kf, $0 \leqslant k < 2m$, *is defined and piecewise-continuous on* $[a, b]$ *with* $M \geqslant 0$ *discontinuities in the points* $\{ y_j\}_{j=1}^M \subset (a, b)$, *assume that $D^k f$ is absolutely continuous on each subinterval (y_i, y_{i+1}), $0 \leqslant i \leqslant M$, (where $y_0 \equiv a$, $y_{M+1} \equiv b$,), with $D^{k+1}f \in L_r[y_i, y_{i+1}]$, $0 \leqslant i \leqslant M$, where $1 \leqslant r \leqslant \infty$. If $\Delta \in \mathscr{P}_\sigma(a, b)$, and if g in $H^{(2m+1)}(\Delta)$ is the unique interpolant of f in the sense of* (3.3), *then for $r \leqslant q \leqslant \infty$,*

$$K\pi^{k-j+(1/q)}\{M^{1/q}\omega(D^k f, \pi) + \pi^{1-(1/r)} \| D^{k+1}f\|_{L_r[a,b]}\} \geqslant \begin{cases} \| D^j(f-g)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j g \|_{L_q[a,b]}, & k < j \leqslant 2m. \end{cases} \tag{3.15}$$

Proof. As in the case of Corollary 3.3, we shall establish the inequality of (3.15) only in the case $q = \infty$; the case of general q with $r \leqslant q \leqslant \infty$ follows again from Jensen's inequality. For any subinterval $[x_i, x_{i+1}]$ determined by the partition Δ of $[a, b]$, suppose first that $[x_i, x_{i+1}]$ contains no discontinuity of $D^k f$. Then, for $0 \leqslant j \leqslant k-1$, it follows from (3.11) with $\alpha = D^k f(x)$ and from (3.13) that

$$\| D^j(f-g)\|_{L_\infty[x_i, x_{i+1}]} \leqslant K\pi^{k+1-j-(1/r)} \| D^{k+1}f\|_{L_r[a,b]} .$$

As in the proof of Lemma 3.2, this inequality is also valid for $k = j$. On the other hand, if $[x_i, x_{i+1}]$ does contain at least one discontinuity of $D^k f$, then it follows from (3.11) with $\alpha = D^k f(x)$ that for $0 \leqslant j \leqslant k$,

$$\| D^j(f-g)\|_{L_\infty[x_i, x_{i+1}]} \leqslant K\pi^{k-j}\omega(D^k f, \pi).$$

Since the sum of the upper bounds of the previous inequalities bounds $\| D^j(f-g)\|_{L_\infty[x_i, x_{i+1}]}$ for *any* i, then we deduce that

$$\| D^j(f-g)\|_{L_\infty[a,b]} \leqslant K\{\pi^{k+1-j-(1/r)} \| D^{k+1}f\|_{L_r[a,b]} + \pi^{k-j}\omega(D^k f, \pi)\},$$

which is the desired result of the first inequality of (3.15) when $q = \infty$. The rest is similarly established. Q.E.D.

As an example illustrating the result of Corollary 3.4, consider any $f \in W_2^1[0, 1]$ with $f(0) = 1$, and define f on $[-1, +1]$ by means of an *odd* extension, so that f is discontinuous at $x = 0$. Then, f satisfies the hypotheses of Corollary 3.4 with $k = 0$ and $M = 1$. As $\omega(f, \pi) \geqslant 2$ and is bounded above, we have from (3.15) with $r = q = 2$ and $j = 0$ that

$$\|f - g\|_{L_2[-1,+1]} \leqslant K\{\pi^{1/2}\omega(f, \pi) + \pi \| Df\|_{L_2[-1,+1]}\}.$$

We now prove an analogue of Lemma 3.1 which includes less smooth functions.

THEOREM 3.5. *Given $f \in C^k[a, b]$ with $0 \leqslant k < 2m$ and given $\Delta \in \mathscr{P}_\sigma(a, b)$, let s be the unique element in* $\mathrm{Sp}(L, \Delta, z)$ *such that for $z_0 \equiv m \equiv z_N$,*

$$\begin{aligned} D^j(f - s)(x_i) &= 0, \qquad 0 \leqslant j \leqslant \min(k, z_i - 1), \quad 0 \leqslant i \leqslant N, \\ D^j s(x_i) &= 0, \qquad \text{if} \quad \min(k, z_i - 1) < j \leqslant z_i - 1, \quad 0 \leqslant i \leqslant N. \end{aligned} \tag{3.16}$$

Then, for $2 \leqslant q \leqslant \infty$,

$$K\pi^{k-j-(1/2)+(1/q)}\{\omega(D^k f, \pi) + \pi^{2m-k} \| f \|_{W_2{}^k[a,b]}\} \geqslant \begin{cases} \| D^j(f - s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_q[a,b]}, & \text{if} \quad k < j \leqslant 2m - 1. \end{cases} \tag{3.17}$$

For polynomial splines, i.e., $L = D^m$, the term involving $\| f \|_{W_2{}^k[a,b]}$ can be deleted in (3.17).

Proof. Given $f \in C^k[a, b]$, let $g \in C^{2m}[a, b]$ be its interpolation in the sense of (3.3) of Lemma 3.2. The triangle inequality gives us for $2 \leqslant q \leqslant \infty$ that

$$\| D^j(f - s)\|_{L_q[a,b]} \leqslant \| D^j(f - g)\|_{L_q[a,b]} + \| D^j(g - s)\|_{L_q[a,b]}, \qquad 0 \leqslant j \leqslant k. \tag{3.18}$$

Next, note that s, while interpolating f as described in (3.16), necessarily also interpolates g in the sense of Lemma 3.1. Thus, from (3.2) of Lemma 3.1,

$$\| D^j(g - s)\|_{L_q[a,b]} \leqslant K\pi^{2m-j-(1/2)+(1/q)} \| g \|_{W_2^{2m}[a,b]}, \qquad 0 \leqslant j \leqslant 2m - 1. \tag{3.19}$$

We now bound $\| g \|_{W_2^{2m}[a,b]}$. For any l with $k < l \leqslant 2m$, we have from (3.4) of Lemma 3.2 that

$$\| D^l g \|_{L_2[a,b]} \leqslant K\pi^{k-l}\omega(D^k f, \pi), \qquad k < l \leqslant 2m. \tag{3.20}$$

For any l with $0 \leqslant l \leqslant k$, we evidently have

$$\| D^l g \|_{L_2[a,b]} \leqslant \| D^l(f - g)\|_{L_2[a,b]} + \| D^l f \|_{L_2[a,b]}, \qquad 0 \leqslant l \leqslant k.$$

Using (3.4) of Lemma 3.2 to bound the first term on the right-hand side of the above inequality gives

$$\| D^l g \|_{L_2[a,b]} \leqslant K\pi^{k-l}\omega(D^k f, \pi) + \| D^l f \|_{L_2[a,b]}, \qquad 0 \leqslant l \leqslant k. \tag{3.21}$$

Then summing the inequalities of (3.20) and (3.21) gives [cf. (2.5)]

$$\| g \|_{W_2^{2m}[a,b]} \equiv \sum_{l=0}^{2m} \| D^l g \|_{L_2[a,b]} \leqslant K\{\pi^{k-2m}\omega(D^k f, \pi) + \| f \|_{W_2{}^k[a,b]}\},$$

and substituting this bound in (3.19) then yields the following upper bound for the last term of (3.18):

$$\| D^j(g - s)\|_{L_q[a,b]} \leqslant K\pi^{k-j-(1/2)+(1/q)}\{\omega(D^k f, \pi) + \pi^{2m-k} \|f\|_{W_2^k[a,b]}\},$$
$$0 \leqslant j \leqslant 2m - 1. \quad (3.22)$$

Of course, if polynomial splines are used ($L = D^m$), the term involving $\|f\|_{W_2^k[a,b]}$ can be deleted (cf. Lemma 3.1). Finally, the first term on the right-hand side of (3.18) can be bounded above by (3.4) of Lemma 3.2, so that combining this bound with (3.22) gives

$$\| D^j(f - s)\|_{L_q[a,b]} \leqslant K\pi^{k-j-(1/2)+(1/q)}\{\omega(D^k f, \pi) + \pi^{2m-k} \|f\|_{W_2^k[a,b]}\},$$
$$0 \leqslant j \leqslant k,$$

the desired result of the first inequality of (3.17). Next, if $k < j \leqslant 2m - 1$, then the use of the same technique to bound the terms on the right-hand side of

$$\| D^j s \|_{L_q[a,b]} \leqslant \| D^j(g - s)\|_{L_q[a,b]} + \| D^j g \|_{L_q[a,b]}, \qquad \text{if} \quad k < j \leqslant 2m - 1,$$

yields the second inequality of (3.17). Q.E.D.

In a similar way, the following extension of Theorem 3.5 is easily established.

COROLLARY 3.6. *With the hypotheses of Theorem* 3.5, *if* $f \in W_r^{k+1}[a, b]$ *with* $1 \leqslant r \leqslant \infty$ *and* $0 \leqslant k < 2m$, *then for* $\max(r, 2) \leqslant q \leqslant \infty$,

$$K\pi^{k+1-j+(1/q)+\min(-1/r,-1/2)} \|f\|_{W_r^{k+1}[a,b]} \geqslant \begin{cases} \| D^j(f - s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_q[a,b]} & \text{if} \quad k < j \leqslant 2m - 1. \end{cases} \quad (3.23)$$

For polynomial splines, $\|f\|_{W_r^{k+1}[a,b]}$, *can be replaced in* (3.23) *by* $\| D^{k+1} f\|_{L_r[a,b]}$.

There is, of course, an obvious extension of the result of Corollary 3.6, under the weaker hypotheses of Corollary 3.4. Such extensions will, for reasons of brevity, be omitted.

We remark that Corollary 3.6 generalizes the result of Lemma 3.1, in that the first inequality of (3.23) of Corollary 3.6 for the case $k = 2m - 1$ and $r = 2$, reduces to the inequality (3.2) of Lemma 3.1.

For the error bounds established in this section for various types of spline interpolation, we remark that in many cases these bounds have been shown (as in Birkhoff, Schultz, and Varga [7], Schultz and Varga [32], and Golomb

[15]) to be sharp, in the sense that the exponents of π cannot in general be improved. Finally, since the result of Corollary 3.6 is valid for general L-splines, we remark that the bounds of (3.23) in particular extend to even less smooth functions all the upper bounds recently obtained by Schultz [31] for polynomial splines, i.e., $L = D^m$.

4. L-Spline Interpolants via Lagrange Interpolation

We begin with the following extension of a result of Schultz [29, Theorem 6.1].

THEOREM 4.1. *Given $f \in C^k[a, b]$, $k \geqslant 0$, and given $\Delta \in \mathscr{P}_\sigma(a, b)$ with at least $(n+1)$ knots, i.e., $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ with $N \geqslant n$, let $L_{n,i}f$, $n \geqslant 1$ fixed, denote the Lagrange polynomial interpolation of degree n of f in the knots $x_i, x_{i+1}, \ldots, x_{i+n}$, where $0 \leqslant i \leqslant N - n$, i.e.,*

$$(L_{n,i}f)(x_j) = f(x_j), \qquad i \leqslant j \leqslant i + n. \tag{4.1}$$

Then, for $s \equiv \min(k, n)$,

$$K\pi^{s-j}\omega(D^s f, \pi) \geqslant \begin{cases} \| D^j(f - L_{n,i}f)\|_{L_\infty[x_i, x_{i+n}]}, & 0 \leqslant j \leqslant s, \\ \| D^j(L_{n,i}f)\|_{L_\infty[x_i, x_{i+n}]}, & j > s. \end{cases} \tag{4.2}$$

Proof. Consider first the case when $k = 0$. It is well known that we can write $L_{n,i}f$ as

$$(L_{n,i}f)(x) = \sum_{j=i}^{i+n} f(x_j)\, l_j(x), \tag{4.3}$$

where

$$l_j(x) \equiv \prod_{\substack{l=i \\ l \neq j}}^{i+n} (x - x_l) \Big/ \prod_{\substack{l=i \\ l \neq j}}^{i+n} (x_j - x_l).$$

From their definition, it is clear that the $l_j(x)$'s form a partition of unity i.e., $\sum_{j=i}^{i+n} l_j(x) \equiv 1$. Thus, we can express f as

$$f(x) = \sum_{j=i}^{i+n} f(x)\, l_j(x), \qquad x \in [x_i, x_{i+n}].$$

Hence,

$$(f - L_{n,i}f)(x) = \sum_{j=i}^{i+n} (f(x) - f(x_j))\, l_j(x), \qquad x \in [x_i, x_{i+n}]. \tag{4.4}$$

Because $\Delta \in \mathscr{P}_\sigma(a, b)$ and m is fixed, it is readily verified that

$$\| D^r l_j(x) \|_{L_\infty[x_i, x_{i+n}]} \leqslant K\pi^{-r}, \qquad r \geqslant 0. \tag{4.5}$$

Using the above inequality for $r = 0$ in (4.4) then gives

$$\| f - L_{n,i} f \|_{L_\infty[x_i, x_{i+n}]} \leqslant K\omega(f, \pi),$$

the special case $s = 0$ of the first inequality of (4.2). Next, since the $l_j(x)$'s form a partition of unity, then for any r with $r > 0$,

$$f(x) \sum_{j=i}^{i+n} D^r l_j(x) \equiv 0, \qquad x \in [x_i, x_{i+n}].$$

Hence, from (4.3) we can write

$$D^r(L_{n,i} f)(x) = \sum_{j=i}^{i+n} (f(x_j) - f(x)) D^r l_j(x), \qquad r > 0, \quad x \in [x_i, x_{i+n}],$$

and applying the bounds of (4.5) in the above expression then gives the second inequality of (4.2) for the special case $s = 0$. For $k > 0$, one can use the Peano kernel theorem argument, as employed in the proof of Lemma 3.2. Q.E.D.

For functions $f \in W_r^{k+1}[a, b]$, we have the following extension of Theorem 4.1.

COROLLARY 4.2. *With the hypotheses of Theorem* 4.1, *if* $f \in W_r^{k+1}[a, b]$ *with* $k \geqslant 0$ *and* $1 \leqslant r \leqslant \infty$, *then for* $s \equiv \min(k, n)$ *and* $r \leqslant q \leqslant \infty$,

$$K\pi^{s+1-j-(1/r)+(1/q)} \| D^{s+1} f \|_{L_r[x_i, x_{i+n}]} \geqslant \begin{cases} \| D^j(f - L_{n,i} f) \|_{L_q[x_i, x_{i+n}]}, & 0 \leqslant j \leqslant s, \\ \| D^j(L_{n,i} f) \|_{L_q[x_i, x_{i+n}]}, & j > s. \end{cases} \tag{4.6}$$

It is now useful to quote a stability (or perturbation) result concerning two-point Hermite interpolation, due to Swartz [37, 38].

LEMMA 4.3. *Let* $h(x)$ *be any polynomial of degree at most* $(4m + 1)$ *satisfying*

$$| D^j h(0) | \leqslant F(\delta) \, \delta^{-j}, \qquad | D^j h(\delta) | \leqslant F(\delta) \cdot \delta^{-j}, \qquad 0 \leqslant j \leqslant 2m, \tag{4.7}$$

for some function $F(\delta)$. *Then, for* $1 \leqslant q \leqslant \infty$,

$$\| D^j h \|_{L_q[0, \delta]} \leqslant KF(\delta) \cdot \delta^{-j+(1/q)}, \qquad j \geqslant 0. \tag{4.8}$$

The next stability result shows that error bounds similar to those of (4.2) of Theorem 4.1 hold for the Lagrange polynomial interpolation of *approximate* data.

THEOREM 4.4. *Given $f \in C^k[a, b]$, $k \geqslant 0$, and given $\Delta \in \mathscr{P}_\sigma(a, b)$ with at least $(n+1)$ knots, i.e., $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ with $N \geqslant n$, let $\tilde{L}_{n,i} f$, $n \geqslant 1$ fixed, denote the Lagrange polynomial interpolation of approximate values of f in the knots $x_i, x_{i+1}, \ldots, x_{i+n}$, $0 \leqslant i \leqslant N-n$, i.e., $\tilde{L}_{n,i} f$ is a polynomial of degree n satisfying*

$$(\tilde{L}_{n,i} f)(x_j) = \alpha_j, \qquad i \leqslant j \leqslant i+n, \tag{4.9}$$

where, with $s \equiv \min(k, n)$, we assume that a function $F(f, \pi)$ exists such that

$$|\alpha_j - f(x_j)| \leqslant \pi^s F(f, \pi), \qquad i \leqslant j \leqslant i+n. \tag{4.10}$$

Then,

$$K\pi^{s-j}\{\omega(D^k f, \pi) + F(f, \pi)\} \geqslant \begin{cases} \| D^j(f - \tilde{L}_{n,i} f)\|_{L_\infty[x_i, x_{i+n}]}, & 0 \leqslant j \leqslant s, \\ \| D^j(\tilde{L}_{n,i} f)\|_{L_\infty[x_i, x_{i+n}]}, & j > s. \end{cases} \tag{4.11}$$

Proof. If $L_{n,i} f$ denotes the Lagrange polynomial interpolation of f in the sense of (4.1) of Theorem 4.1, then we can write

$$\begin{aligned} D^j(f - \tilde{L}_{n,i} f)(x) &= D^j(f - L_{n,i} f)(x) + D^j(L_{n,i} f - \tilde{L}_{n,i} f)(x), \\ D^j \tilde{L}_{n,i} f(x) &= D^j(L_{n,i} f)(x) + D^j(\tilde{L}_{n,i} f - L_{n,i} f)(x), \end{aligned} \tag{4.12}$$

where $\tilde{L}_{n,i} f$ is the Lagrange polynomial interpolation defined by (4.9). Clearly,

$$\begin{aligned} &\| D^j(f - \tilde{L}_{n,i} f)\|_{L_\infty[x_i, x_{i+n}]} \\ &\qquad \leqslant \| D^j(f - L_{n,i} f)\|_{L_\infty[x_i, x_{i+n}]} + \| D^j(L_{n,i} f - \tilde{L}_{n,i} f)\|_{L_\infty[x_i, x_{i+n}]}. \end{aligned} \tag{4.13}$$

For $0 \leqslant j \leqslant s$, the first error bound of (4.2) of Theorem 4.1 can be used to bound the first term on the right of (4.13). To bound the second term on the right of (4.13), we have from the definition of $l_j(x)$ in (4.3) and the hypothesis of (4.10) that

$$\begin{aligned} | D^j(L_{n,i} f - \tilde{L}_{n,i} f)(x)| &= \left| \sum_{k=i}^{i+n} (f(x_k) - \alpha_k)\, D^j l_k(x) \right| \\ &\leqslant \pi^s F(f, \pi) \sum_{k=i}^{i+n} | D^j l_k(x)| \\ &\leqslant K\pi^{s-j} F(f, \pi), \qquad x \in [x_i, x_{i+n}], \end{aligned}$$

the last inequality following from (4.5). Thus,

$$\| D^j(L_{n,i}f - \tilde{L}_{n,i}f)\|_{L_\infty[x_i, x_{i+n}]} \leqslant K\pi^{s-j}F(f, \pi), \tag{4.14}$$

which then gives the desired result of the first inequality of (4.11). The proof of the second inequality of (4.11) is similar. Q.E.D.

The analogue of Corollary 4.2 for Theorem 4.4 is

COROLLARY 4.5. *With the hypotheses of Theorem* 4.4, *if* $f \in W_r^{k+1}[a, b]$ *with* $k \geqslant 0$ *and* $1 \leqslant r \leqslant \infty$, *and for* $s \equiv \min(k, n)$, *if* [*cf*, (4.10)]

$$| \alpha_j - f(x_j)| \leqslant K\pi^{s+1-(1/r)} \| D^{s+1}f \|_{L_r[x_i, x_{i+n}]}, \qquad i \leqslant j \leqslant i + n, \tag{4.15}$$

then for $r \leqslant q \leqslant \infty$,

$$K\pi^{s+1-j-(1/r)+(1/q)} \| D^{s+1}f \|_{L_r[x_i, x_{i+n}]} \geqslant \begin{cases} \| D^j(f - \tilde{L}_{n,i}f)\|_{L_q[x_i, x_{i+n}]}, & 0 \leqslant j \leqslant s, \\ \| D^j(\tilde{L}_{n,i}f)\|_{L_q[x_i, x_{i+n}]}, & j > s. \end{cases} \tag{4.16}$$

We now show how one may estimate derivatives of f at knots, needed to define L-spline interpolants.

THEOREM 4.6. *Given* $f \in C^k[a, b]$ *with* $0 \leqslant k < 2m$ *and given* $\Delta \in \mathscr{P}_\sigma(a, b)$ *with at least* $2m$ *knots, let* s *be the unique element in* $\mathrm{Sp}(L, \Delta, z)$ *such that* $z_0 \equiv m \equiv z_N$,

$$D^js(x_i) = D^j(L_{2m-1,i}f)(x_i), \qquad 0 \leqslant j \leqslant z_i - 1, 0 \leqslant i \leqslant N, \tag{4.17}$$

where $L_{2m-1,i}f$ *is any Lagrange polynomial interpolation of* f *in* $2m$ *consecutive knots* $x_j, x_{j+1}, \ldots, x_{j+2m-1}$ *with* $x_i \in [x_j, x_{j+2m-1}]$. *Then, for* $2 \leqslant q \leqslant \infty$,

$$K\pi^{k-j-(1/2)+(1/q)}\{\omega(D^kf, \pi) + \pi^{2m-k} \| f \|_{W_2{}^k[a,b]}\} \geqslant \begin{cases} \| D^j(f - s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^js \|_{L_q[a,b]}, & \text{if } k < j \leqslant 2m - 1. \end{cases} \tag{4.18}$$

For polynomial splines, i.e., $L = D^m$, *the term involving* $\|f\|_{W_2{}^k[a,b]}$ *in* (4.18) *can be deleted.*

Proof. This is a "3ε" proof. Let $t \in \mathrm{Sp}(L, \Delta, z)$ be the unique L-spline which interpolates f in the sense of (3.16) of Theorem 3.5, and let $h \in H^{(2m+)1}(\Delta)$ be the unique polynomial spline such that

$$D^jh(x_i) = \begin{cases} D^j(f - L_{2m-1,i}f)(x_i), & 0 \leqslant j \leqslant k, \\ -D^j(L_{2m-1,i}f)(x_i), & k < j \leqslant 2m. \end{cases} \tag{4.19}$$

Writing $f - s = (f - t) + h + [(t - s) - h]$, we evidently have that

$$\| D^j(f - s)\|_{L_q[a,b]}$$
$$\leqslant \| D^j(f - t)\|_{L_q[a,b]} + \| D^j h \|_{L_q[a,b]} + \| D^j(h - [t - s])\|_{L_q[a,b]} \quad (4.20)$$

for $0 \leqslant j \leqslant k$. The first term on the right side of (4.20) can be bounded above by (3.17) of Theorem 3.5. For the second term on the right side of (4.20), we see from the definition in (4.19) and Theorem 4.1 that h satisfies the conditions (4.7) of Lemma 4.3 on each subinterval $[x_i, x_{i+1}]$ defined by Δ, with

$$F(\delta) \equiv K\delta^k \omega(D^k f, \delta).$$

Again, because $\Delta \in \mathscr{P}_\sigma(a, b)$, it follows from (4.8) of Lemma 4.3 that for $2 \leqslant q \leqslant \infty$,

$$\| D^j h \|_{L_q[a,b]} \leqslant K\pi^{k-j}\omega(D^k f, \pi), \qquad 0 \leqslant j \leqslant 2m. \quad (4.21)$$

For the third term on the right side of (4.20), we observe that $t - s$ is the unique L-spline interpolant of $h \in C^{2m}[a, b]$, in the sense of (3.1). As such, we can apply the bounds of (3.2) of Lemma 3.1. But, from (4.21), we see that

$$\| h \|_{W_2^{2m}[a,b]} \leqslant K\pi^{k-2m}\omega(D^k f, \pi).$$

Consequently, from (3.2), for $2 \leqslant q \leqslant \infty$,

$$\| D^j(h - (t - s))\|_{L_q[a,b]} \leqslant K\pi^{k-j-(1/2)+(1/q)}\omega(D^k f, \pi), \qquad 0 \leqslant j \leqslant 2m - 1.$$

Combining these bounds gives the desired result of the first inequality of (4.18). The remainder of the proof is similarly established. Q.E.D.

We remark that the error bounds of (4.18) of Theorem 4.6 and (3.17) of Theorem 3.5 are *identical* even though the splines of interpolation in each case are necessarily different. Note moreover that the interpolation of (4.17) of Theorem 4.6 does *not* explicitly depend upon the continuity class of f, in contrast with the interpolation of (3.16) of Theorem 3.5. In Section 5, we shall show in a stability result that there is in fact a family of splines approximately interpolating f which satisfy these identical error bounds.

We conclude this section with an analogue of Corollary 3.6.

COROLLARY 4.7. *With the hypotheses of Theorem* 4.6, *if* $f \in W_r^{k+1}[a, b]$ *with* $1 \leqslant r \leqslant \infty$ *and* $0 \leqslant k < 2m$, *then for* $\max(r, 2) \leqslant q \leqslant \infty$,

$$K\pi^{k+1-j+(1/q)+\min(-1/r,-1/2)} \| f \|_{W_r^{k+1}[a,b]}$$
$$\geqslant \begin{cases} \| D^j(f - s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_q[a,b]}, & \text{if } k < j \leqslant 2m - 1. \end{cases}$$

For polynomial splines, $\| f \|_{W_r^{k+1}[a,b]}$ *can be replaced by* $\| D^{k+1} f \|_{L^r[a,b]}$.

5. Stability of L-Spline Interpolation

The technique of proof of Theorem 4.6 yields the following stability result of Theorem 5.1, which generalizes Theorem 4.6. Roughly stated, one can suitably perturb the data defining an L-spline interpolant without affecting the nature of the original error bounds.

Theorem 5.1. *Given $f \in C^k[a, b]$ with $0 \leqslant k < 2m$ and given $\Delta \in \mathscr{P}_\sigma(a, b)$, let s be the unique element in* $\mathrm{Sp}(L, \Delta, z)$ *such that for $z_0 = m = z_N$,*

$$D^j s(x_i) = \alpha_{i,j}, \qquad 0 \leqslant j \leqslant z_i - 1, \qquad 0 \leqslant i \leqslant N, \tag{5.1}$$

where it is assumed that functions $F_i(f, \pi)$ exist such that for $0 \leqslant i \leqslant N$,

$$K\pi^{k-j} F_i(f, \pi) \geqslant \begin{cases} | D^j f(x_i) - \alpha_{i,j} |, & 0 \leqslant j \leqslant \min(k, z_i - 1), \\ | \alpha_{i,j} |, & \textit{if} \;\; \min(k, z_i - 1) < j \leqslant z_i - 1. \end{cases} \tag{5.2}$$

Define $\| F \|_r \equiv [\pi \sum_{i=0}^{N} F_i^r(f, \pi)]^{(1/r)}$, for $1 \leqslant r \leqslant \infty$. Then, for $2 \leqslant q \leqslant \infty$,

$$K\pi^{k-j-(1/2)+(1/q)} \{(\omega(D^k f, \pi) + \| F \|_2 + \pi^{2m-k} \| f \|_{W_2^k[a,b]}\}$$
$$\geqslant \begin{cases} \| D^j(f - s) \|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_q[a,b]}, & \textit{if} \;\; k < j \leqslant 2m - 1. \end{cases} \tag{5.3}$$

For polynomial splines, i.e., $L = D^m$, the term involving $\| f \|_{W_2^k[a,b]}$ in (5.3) can be deleted.

Proof. Let $h \in H^{(2m+1)}(\Delta)$ be the unique polynomial spline such that

$$D^j h(x_i) = \begin{cases} D^j f(x_i) - \alpha_{i,j}, & 0 \leqslant j \leqslant \min(k, z_i - 1), \\ -\alpha_{i,j}, & \text{if} \;\; \min(k, z_i - 1) < j \leqslant z_i - 1, \\ 0, & z_i - 1 < j \leqslant 2m, \end{cases} \tag{5.4}$$

for all $0 \leqslant i \leqslant N$, and let $t \in \mathrm{Sp}(L, \Delta, z)$ be the unique L-spline defined by (3.16) of Theorem 3.5. Writing $f - s = (f - t) + h + [(t - s) - h]$, then for $2 \leqslant q \leqslant \infty$, and $0 \leqslant j \leqslant k$,

$$\| D^j(f - s) \|_{L_q[a,b]}$$
$$\leqslant \| D^j(f - t) \|_{L_q[a,b]} + \| D^j h \|_{L_q[a,b]} + \| D^j(h - [t - s]) \|_{L_q[a,b]}. \tag{5.5}$$

The first term on the right side of (5.5) can be bounded above by (3.17) of Theorem 3.5, i.e.,

$$\| D^j(f - t) \|_{L_q[a,b]} \leqslant K\{\pi^{k-j-(1/2)+(1/q)} \omega(D^k f, \pi) + \pi^{2m-j-(1/2)+(1/q)} \| f \|_{W_2^k[a,b]}\}.$$

Next, from the hypotheses of (5.2), and the definitions of (5.4), we evidently have from (4.8) of Lemma 4.3 that

$$| D^j h(x)| \leqslant K\pi^{k-j} \max(F_i(f, \pi); F_{i+1}(f, \pi)), \qquad x \in [x_i , x_{i+1}]. \tag{5.6}$$

But, since

$$\max(F_i(f, \pi); F_{i+1}(f, \pi))$$
$$\leqslant (F_i^2(f, \pi) + F_{i+1}^2(f, \pi))^{1/2} \leqslant \sum_{j=0}^{N} F_j^2(f, \pi))^{1/2} = \pi^{-1/2} \| F \|_2$$

for *all* $0 \leqslant i \leqslant N - 1$, then

$$\| D^j h \|_{L_\infty[a,b]} \leqslant K\pi^{k-j-1/2} \| F \|_2 .$$

Similarly, squaring, integrating the inequality of (5.6), and then summing on i yields

$$\| D^j h \|_{L_2[a,b]} \leqslant K\pi^{k-j} \| F \|_2 . \tag{5.7}$$

Then, it follows from the last two inequalities for $2 \leqslant q \leqslant \infty$ that

$$\| D^j h \|_{L_q[a,b]} \leqslant K\pi^{k-j-(1/2)+(1/q)} \| F \|_2 , \tag{5.8}$$

which bounds the second term on the right of (5.5). Next, we observe, as in the proof of Theorem 4.4, that $t - s$ is the unique L-spline interpolation of $h \in C^{2m}[a, b]$ in the sense of (3.1). Since we have from (5.7) that

$$\| h \|_{W_2^{2m}[a,b]} \leqslant K\pi^{k-2m} \| F \|_2 ,$$

the error bounds of (3.2) of Lemma 3.1 then give us

$$\| D^j(h - (t - s))\|_{L_q[a,b]} \leqslant K\pi^{k-j-(1/2)+(1/q)} \| F \|_2 .$$

Inserting these inequalities in (5.5) then gives us the first desired inequality of (5.3). The lower inequality of (5.3) follows in similar fashion. Q.E.D.

We remark that the error bounds of (5.3) of Theorem 5.1 reduce to the common bounds of (3.17) of Theorem 3.5 and (4.18) of Theorem 4.6 whenever $\| F \|_2 \leqslant K\omega(D^k f, \pi)$. Thus, there is a *family* of splines approximately interpolating f which satisfy the bounds (3.17) or (4.18).

We conclude this section with an analogue of Corollary 4.7.

COROLLARY 5.2. *With the hypotheses of Theorem* 5.1, *if* $f \in W_r^{k+1}[a, b]$

with $1 \leqslant r \leqslant \infty$ *and* $0 \leqslant k < 2m$, *and if functions* $F_i(f, \pi)$ *exist* [*cf.* (5.2)] *such that for* $0 \leqslant i \leqslant N$,

$$K\pi^{k+1-j}F_i(f, \pi) \geqslant \begin{cases} | D^j f(x_i) - \alpha_{i,j} |, & 0 \leqslant j \leqslant \min(k, z_i - 1), \\ | \alpha_{i,j} |, & \text{if} \ \min(k, z_i - 1) < j \leqslant z_i - 1, \end{cases} \tag{5.9}$$

then for $\max(r, 2) \leqslant q \leqslant \infty$,

$$K\pi^{k+1-j+(1/q)+\min(-1/r,-1/2)}\{\| f \|_{W_r^{k+1}[a,b]} + \| F \|_r\}$$

$$\geqslant \begin{cases} \| D^j(f - s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_q[a,b]}, & \text{if} \ k < j \leqslant 2m - 1. \end{cases}$$

For polynomial splines, $\| f \|_{W_r^{k+1}[a,b]}$ *can be replaced by* $\| D^{k+1} f \|_{L_r[a,b]}$.

6. Improved Error Bounds for Interpolating Hermite L-Splines

Thus far, our error bounds for general L-spline interpolation in the *uniform* norm [cf. (3.23) with $q = \infty$] for functions f in $W_2^{k+1}[a, b]$ have had an exponent of π which is one-half less than those for the corresponding polynomial spline error bounds in the uniform norm [cf. (7.17)] for functions f in $W_\infty^{k+1}[a, b]$. In general, these error bounds for L-spline interpolation are sharp (cf. Schultz and Varga [32]), but we now show that the general improved form of the error bounds in the uniform norm for polynomial splines can also be obtained for what we shall call *Hermite L-spline interpolation.* If the incidence vector $\hat{z} = (\hat{z}_1, \ldots, \hat{z}_{N-1})$ for the L-spline space $\mathrm{Sp}(L, \Delta, \hat{z})$ is such that $\hat{z}_i \equiv m$, $1 \leqslant i \leqslant N - 1$, then

$$H(L, \Delta) \equiv \mathrm{Sp}(L, \Delta, \hat{z})$$

is called the *Hermite L-spline space*, where L is the differential operator of order m of (2.3).

The first result (6.2) of this section involving Hermite L-splines improves (3.17) of Theorem 3.5 for any $2 < q \leqslant \infty$.

Theorem 6.1. *Given* $f \in C^k[a, b]$ *with* $0 \leqslant k < 2m$ *and given* $\Delta \in \mathscr{P}_\sigma(a, b)$, *let* s *be the unique element in* $H(L, \Delta)$ *such that*

$$\begin{aligned} D^j(f - s)(x_i) &= 0, \qquad 0 \leqslant j \leqslant \min(k, m - 1), \quad 0 \leqslant i \leqslant N, \\ D^j s(x_i) &= 0, \quad \text{if} \ \min(k, m - 1) < j \leqslant m - 1, \quad 0 \leqslant i \leqslant N. \end{aligned} \tag{6.1}$$

Then,

$$K\pi^{k-j}\{\omega(D^k f, \pi) + \pi^{2m-k} \| f \|_{W_\infty{}^k[a,b]}\} \geqslant \begin{cases} \| D^j(f-s)\|_{L_\infty[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_\infty[a,b]}, & \text{if } \; k < j \leqslant 2m-1. \end{cases} \tag{6.2}$$

For polynomial splines, the term involving $\| f \|_{W_\infty{}^k[a,b]}$ *can be deleted in* (6.2).

Proof. Let $g \in H^{(2m+1)}(\Delta)$ be the polynomial spline interpolation of f in the sense of (3.3) of Lemma 3.2. Writing $f - s = (f - g) + (g - s)$, and $s = g + (s - g)$, it is evident that

$$\begin{aligned} \| D^j(f-s)\|_{L_\infty[a,b]} &\leqslant \| D^j(f-g)\|_{L_\infty[a,b]} + \| D^j(g-s)\|_{L_\infty[a,b]}, \\ &\qquad\qquad 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_\infty[a,b]} &\leqslant \| D^j g \|_{L_\infty[a,b]} + \| D^j(g-s)\|_{L_\infty[a,b]}, \\ &\qquad\qquad k < j \leqslant 2m-1. \end{aligned} \tag{6.3}$$

Clearly, the results of Lemma 3.2 suitably bound the first terms on the right side of (6.3). Next, by definition, s is also the $H(L, \Delta)$-interpolant of g in the sense that

$$D^j(g-s)(x_i) = 0, \qquad 0 \leqslant j \leqslant m-1, \qquad 0 \leqslant i \leqslant N.$$

As such, we could make use of recent results of Ciarlet and Varga [13, Corollary 4], but a self-contained proof is instead given. Let $G(\xi, \eta)$ denote the Green's function for the boundary-value problem

$$\begin{aligned} D^{2m}y(x) &= \tau(x),\ x \in (0, 1), \\ D^j y(0) &= D^j y(1) = 0,\ 0 \leqslant j \leqslant m-1. \end{aligned}$$

Because s interpolates g at the knots x_i we can write for $x \in [x_i, x_{i+1}]$ that

$$D^j(g-s)(x) = h_i^{2m-j} \int_0^1 \left[\partial^j G\left(\frac{x - x_i}{h_i}, \eta\right)\Big/\partial\xi^j\right] D^{2m}(g-s)(x_i + \eta h_i)\, d\eta, \qquad 0 \leqslant j \leqslant 2m-1,$$

where $h_i \equiv x_{i+1} - x_i$; and the boundedness of $\partial^j G(\xi, \eta)/\partial\xi^j$ in $[0, 1] \times [0, 1]$ gives us for $0 \leqslant j \leqslant 2m-1$, $x \in [x_i, x_{i+1}]$, that

$$\begin{aligned} | D^j(g-s)(x)| &\leqslant Kh_i^{2m-j} \| D^{2m}(g-s)\|_{L_\infty[x_i,x_{i+1}]} \\ &\leqslant Kh_i^{2m-j} \| D^{2m} g \|_{L_\infty[x_i,x_{i+1}]} + Kh_i^{2m-j} \| D^{2m} s \|_{L_\infty[x_i,x_{i+1}]}. \end{aligned} \tag{6.4}$$

Next, since the leading coefficient $c_m(x)$ in the definition of the operator L is strictly positive in $[a, b]$ (cf. Section 2), then $L^*Ls(x) = 0$ in (x_i, x_{i+1}) implies that

$$D^{2m}s(x) = \sum_{l=0}^{2m-1} d_l(x)\, D^l s(x),$$

where each $d_l(x)$ is bounded on $[a, b]$, so that

$$\| D^{2m}s \|_{L_\infty[a,b]} \leqslant K \sum_{l=0}^{2m-1} \| D^l s \|_{L_\infty[a,b]} .$$

By means of the triangle inequality, this can also be bounded above by

$$\| D^{2m}s \|_{L_\infty[a,b]} \leqslant K \sum_{l=0}^{k} \{\| D^l f \|_{L_\infty[a,b]} + \| D^l(f - g)\|_{L_\infty[a,b]}$$
$$+ \| D^l(g - s)\|_{L_\infty[a,b]}\} + K \sum_{l=k+1}^{2m-1} \| D^l s \|_{L_\infty[a,b]} .$$

The first sum, by (2.5), is just $K \| f \|_{W_\infty{}^k[a,b]}$, and the second sum is bounded above from (3.4) of Lemma 3.2 by $K\omega(D^k f, \pi)$, which in turn is bounded above by $K \| f \|_{W_\infty{}^k[a,b]}$. The third and fourth sums are, from (3.23) of Corollary 3.6 with $k = 2m - 1$ and $r = \infty$, bounded above by $\| g \|_{W_\infty^{2m}[a,b]}$, and hence

$$\| D^{2m}s \|_{L_\infty[a,b]} \leqslant K\{\| f \|_{W_\infty{}^k[a,b]} + \| g \|_{W_\infty^{2m}[a,b]}\}.$$

Thus, comparing with (6.4), it remains to bound $\| g \|_{W_\infty^{2m}[a,b]}$. By definition, we can write

$$\| g \|_{W_\infty^{2m}[a,b]} = \sum_{j=0}^{k} \| D^j g \|_{L_\infty[a,b]} + \sum_{j=k+1}^{2m} \| D^j g \|_{L_\infty[a,b]} ,$$

and hence, from the triangle inequality,

$$\| g \|_{W_\infty^{2m}[a,b]} \leqslant \sum_{j=0}^{k} \{\| D^j(f - g)\|_{L_\infty[a,b]} + \| D^j f \|_{L_\infty[a,b]}\} + \sum_{j=k+1}^{2m} \| D^j g \|_{L_\infty[a,b]} .$$

Applying the bounds of (3.4) of Lemma 3.2, it follows then that

$$\| g \|_{W_\infty^{2m}[a,b]} \leqslant \| f \|_{W_\infty{}^k[a,b]} + K\pi^{k-2m}\omega(D^k f, \pi).$$

But combining this with the inequality of (6.4) gives for $0 \leqslant j \leqslant 2m - 1$

$$\| D^j(g - s)\|_{L_\infty[a,b]} \leqslant K\{\pi^{k-j}\omega(D^k f, \pi) + \pi^{2m-j} \| f\|_{W_\infty{}^k[a,b]}\},$$

from which the inequalities of (6.2) follow from (6.3). Q.E.D.

COROLLARY 6.2. *With the hypotheses of Theorem* 6.1, *if* $f \in W_r^{k+1}[a, b]$ *with* $1 \leqslant r \leqslant \infty$ *and* $0 \leqslant k < 2m$, *then for* $\max(r, 2) \leqslant q \leqslant \infty$,

$$K\pi^{k+1-j-(1/r)+(1/q)} \| f\|_{W_r^{k+1}[a,b]} \geqslant \begin{cases} \| D^j(f - s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_q[a,b]}, & \text{if } \; k < j \leqslant 2m - 1. \end{cases} \tag{6.5}$$

The technique of proof of Theorem 6.1 yields the following stability result of Theorem 6.3. We omit its proof.

THEOREM 6.3. *Given* $f \in C^k[a, b]$ *with* $0 \leqslant k < 2m$ *and given* $\Delta \in \mathscr{P}_o(a, b)$, *let s be the unique element in* $H(L, \Delta)$ *such that*

$$D^j s(x_i) = \alpha_{i,j}, \qquad 0 \leqslant j \leqslant m - 1, 0 \leqslant i \leqslant N, \tag{6.6}$$

where it is assumed that functions $F_i(f, \pi)$ *exist such that for* $0 \leqslant i \leqslant N$,

$$K\pi^{k-j}F_i(f, \pi) \geqslant \begin{cases} | D^j f(x_i) - \alpha_{i,j} |, & 0 \leqslant j \leqslant \min(k, m - 1), \\ | \alpha_{i,j} |, & \text{if } \; \min(k, m - 1) < j \leqslant m - 1. \end{cases} \tag{6.7}$$

Then,

$$K\{\pi^{k-j}(\omega(D^k f, \pi) + \| F\|_\infty + \pi^{2m-k} \| f\|_{W_\infty{}^k[a,b]}\}$$
$$\geqslant \begin{cases} \| D^j(f - s)\|_{L_\infty[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j f\|_{L_\infty[a,b]}, & \text{if } \; k < j \leqslant 2m - 1. \end{cases} \tag{6.8}$$

In particular, if

$$\alpha_{i,j} \equiv D^j(L_{2m-1,i} f)(x_i), \qquad 0 \leqslant j \leqslant m - 1, 0 \leqslant i \leqslant N, \tag{6.9}$$

where $L_{2m-1,i} f$ *is the Lagrange polynomial interpolation of* f *in the knots* $x_j, x_{j+1}, \ldots, x_{j+2m-1}$ *with* $x_i \in [x_j, x_{j+2m-1}]$, *then the bounds of* (6.8) *are valid with the term* $\| F\|_\infty$ *deleted.*

COROLLARY 6.4. *With the hypotheses of Theorem* 6.3, *if* $f \in W_r^{k+1}[a, b]$ *with* $1 \leqslant r \leqslant \infty$ *and* $0 \leqslant k < 2m$ *and given* $\Delta \in \mathscr{P}_o(a, b)$, *let s be the unique* *element in* $H(L, \Delta)$ *satisfying* (6.6), *where it is assumed that functions* $F_i(f, \pi)$ *exist* [*cf.* (6.7)] *such that for* $0 \leqslant i \leqslant N$,

$$K\pi^{k+1-j}F_i(f, \pi) \geqslant \begin{cases} | D^j f(x_i) - \alpha_{i,j} |, & 0 \leqslant j \leqslant \min(k, m-1), \\ | \alpha_{i,j} |, & \text{if } \min(k, m-1) < j \leqslant m-1, \end{cases} \tag{6.10}$$

then for $\max(r, 2) \leqslant q \leqslant \infty$,

$$K\pi^{k+1-j-(1/r)+(1/q)}\{\| f \|_{W_r^{k+1}[a,b]} + \| F \|_r\} \geqslant \begin{cases} \| D^j(f-s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_q[a,b]}, & \text{if } k < j \leqslant 2m-1. \end{cases} \tag{6.11}$$

7. Polynomial Spline Interpolation over Uniform Partitions

In this section, we prove a result for smooth polynomial spline interpolation over *uniform* partitions. We begin by stating the following result of Swartz [37], involving "odd-derivative" boundary interpolation [cf. (7.2)].

Lemma 7.1. *Given* $f \in C^{2m}[a, b]$ *and given* $\Delta_u \in \mathscr{P}_1(a, b)$, *let* s *be the unique element in* $\mathrm{Sp}^{(m)}(\Delta_u)$ *such that*

$$\begin{gathered} (f-s)(x_i) = 0, \qquad 0 \leqslant i \leqslant N, \\ D^{2j-1}(f-s)(a) = D^{2j-1}(f-s)(b) = 0, \\ 1 \leqslant j \leqslant m-1, \qquad \text{if } m > 1. \end{gathered} \tag{7.1}$$

Then,

$$\| D^j(f-s)\|_{L_\infty[a,b]} \leqslant K\pi^{2m-j} \| D^{2m} f \|_{L_\infty[a,b]}, \qquad 0 \leqslant j \leqslant 2m-1. \tag{7.2}$$

The result of Lemma 7.1 can be generalized as follows; we remark that the case $k = 0$ can be found in Swartz [38, Corollary 9.1].

Lemma 7.2. *Given* $f \in C^k[a, b]$ *with* $0 \leqslant k < 2m$ *and given* $\Delta_u \in \mathscr{P}_1(a, b)$, *let* s *be the unique element in* $\mathrm{Sp}^{(m)}(\Delta_u)$ *such that*

$$\begin{gathered} (f-s)(x_i) = 0, \qquad 0 \leqslant i \leqslant N, \\ D^{2j-1}(f-s)(a) = D^{2j-1}(f-s)(b) = 0 \qquad \text{if } 1 \leqslant 2j-1 \leqslant k, \\ D^{2j-1}s(a) = D^{2j-1}s(b) = 0 \qquad \text{if } k < 2j-1 \leqslant 2m-3. \end{gathered} \tag{7.3}$$

Then,

$$K\pi^{k-j}\omega(D^k f, \pi) \geqslant \begin{cases} \| D^j(f-s)\|_{L_\infty[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_\infty[a,b]}, & \text{if } k < j \leqslant 2m-1. \end{cases} \tag{7.4}$$

Proof. Let $g \in H^{(2m+1)}(\Delta_u)$ be the unique interpolation of f in the sense of (3.3) of Lemma 3.2. Then, s, as defined in (7.3), is the unique $\mathrm{Sp}^{(m)}(\Delta_u)$-interpolant of g in the sense of (7.1) of Lemma 7.1, and hence, from (7.2),

$$\| D^j(g - s)\|_{L_\infty[a,b]} \leqslant K\pi^{2m-j} \| D^{2m}g \|_{L_\infty[a,b]}, \qquad 0 \leqslant j \leqslant 2m - 1.$$

But, from (3.4) of Lemma 3.2, $\| D^{2m}g \|_{L_\infty[a,b]} \leqslant K\pi^{k-2m}\omega(D^k f, \pi)$ so that

$$\| D^j(g - s)\|_{L_\infty[a,b]} \leqslant K\pi^{k-j}\omega(D^k f, \pi), \qquad 0 \leqslant j \leqslant 2m - 1. \quad (7.5)$$

On the other hand, from (3.4) of Lemma 3.2,

$$\| D^j(f - g)\|_{L_\infty[a,b]} \leqslant K\pi^{k-j}\omega(D^k f, \pi), \qquad 0 \leqslant j \leqslant k,$$

and combining the last two inequalities establishes the first inequality of (7.4). The second inequality of (7.4) follows similarly from (7.5) and (3.4) of Lemma 3.2. Q.E.D.

We remark that the inequality of (7.2) will be seen (Lemma 8.3) to be valid also using L_2 norms throughout. Hence, the conclusion of (7.2) can be strengthened to read

$$\| D^j(f - s)\|_{L_q[a,b]} \leqslant K\pi^{2m-j} \| D^{2m}f \|_{L_q[a,b]},$$
$$0 \leqslant j \leqslant 2m - 1, \qquad 2 \leqslant q \leqslant \infty. \quad (7.2')$$

This is used in proving

COROLLARY 7.3. *With the hypotheses of Lemma* 7.2, *if* $f \in W_r^{k+1}[a, b]$ *with* $0 \leqslant k < 2m$ *and* $1 \leqslant r \leqslant \infty$, *then for* $\max(r, 2) \leqslant q \leqslant \infty$,

$$K\pi^{k+1-j-(1/r)+(1/q)} \| D^{k+1}f \|_{L_r[a,b]} \geqslant \begin{cases} \| D^j(f - s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_q[a,b]}, & \text{if } \ k < j \leqslant 2m - 1. \end{cases} \quad (7.6)$$

We now prove one of our main results, which for spline spaces improves (3.17) of Theorem 3.5 for any $2 < q \leqslant \infty$.

THEOREM 7.4. *Given* $f \in C^k[a, b]$ *with* $0 \leqslant k < 2m$ *and given* $\Delta_u \in \mathscr{P}_1(a, b)$, *let s be the unique element in* $\mathrm{Sp}^{(m)}(\Delta_u)$ *such that*

$$\begin{aligned} (f - s)(x_i) &= 0, \qquad 0 \leqslant i \leqslant N, \\ D^j(f - s)(a) &= D^j(f - s)(b) = 0 \qquad \text{for } \ 0 \leqslant j \leqslant \min(k, m - 1), \\ D^j s(a) &= D^j s(b) = 0 \qquad \text{if } \min(k, m - 1) < j \leqslant m - 1. \end{aligned} \quad (7.7)$$

Then,

$$K\pi^{k-j}\omega(D^k f, \pi) \geqslant \begin{cases} \| D^j(f-s)\|_{L_\infty[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_\infty[a,b]}, & \text{if } k < j \leqslant 2m-1. \end{cases} \tag{7.8}$$

Moreover, if Δ_u has at least 2m knots, and $\tilde{s} \in \mathrm{Sp}^{(m)}(\Delta_u)$ is instead defined by means of

$$\begin{aligned} (f-\tilde{s})(x_i) &= 0, \qquad 0 \leqslant i \leqslant N, \\ D^j\tilde{s}(a) &= D^j(L_{2m-1,0}f)(a), \qquad 0 \leqslant j \leqslant m-1, \end{aligned} \tag{7.9}$$

where $L_{2m-1,0}f$ is the Lagrange polynomial interpolation of f in the knots $x_0, x_1, ..., x_{2m-1}$, with a similar definition of $\tilde{s}$ at $x = b$, then the bounds of (7.8) *remain unchanged if s is replaced by $\tilde{s}$.*

Proof.[1] Let $t \in \mathrm{Sp}^{(m)}(\Delta_u)$ be the unique polynomial spline which interpolates f in the sense of (7.3) of Lemma 7.2. We now construct a particular polynomial spline $h \in H^{(2m+1)}(\Delta_u)$. To begin the construction, let

$$\{\phi_{0,j}(x), \phi_{1,j}(x)\}_{j=0}^{2m}$$

be the unique polynomials of degree $(4m+1)$ such that

$$D^l\phi_{i,j}(0) = \delta_{l,j}\delta_{i,0}, \quad D^l\phi_{i,j}(1) = \delta_{l,j}\delta_{i,1}, \quad 0 \leqslant l, j \leqslant 2m, \ i = 0, 1.$$

Explicitly, for example, we have

$$\phi_{0,0}(x) = \int_x^1 t^{2m}(1-t)^{2m}\,dt \Big/ \int_0^1 t^{2m}(1-t)^{2m}\,dt.$$

Next, we define for $0 \leqslant j \leqslant 2m$

$$\tilde{\phi}_{0,j}(x) = \begin{cases} \pi^j\phi_{0,j}((x-a)/\pi), & x \in [a, a+\pi], \\ 0, & \text{otherwise}, \end{cases}$$

$$\tilde{\phi}_{1,j}(x) = \begin{cases} \pi^j\phi_{1,j}((x-b+\pi)/\pi), & x \in [b-\pi, b], \\ 0, & \text{otherwise}. \end{cases}$$

[1] We have just discovered a recent result of Subbotin [42, Corollary 2], showing that there exists a spline of odd (even) degree, interpolating f at (half-way between) the interior knots, which satisfies the first inequality of (7.8). The boundary conditions and the proofs, however, are different.

By construction, each $\tilde{\phi}_{i,j}(x)$ is a polynomial spline in $H^{(2m+1)}(\Delta_u)$. We then set

$$h(x) \equiv \sum_{j=0}^{\min(k,m-1)} \{D^j(t-s)(a) \cdot \tilde{\phi}_{0,j}(x) + D^j(t-s)(b) \cdot \tilde{\phi}_{1,j}(x)\}$$

$$+ \sum_{j=\min(k,m-1)+1}^{2m-1} \{D^j t(a) \cdot \tilde{\phi}_{0,j}(x) + D^j t(b) \cdot \tilde{\phi}_{1,j}(x)\}, \tag{7.10}$$

so that h is also an element of $H^{(2m+1)}(\Delta_u)$. Writing

$$f - s = (f-t) + h + [(t-s) - h],$$

it is evident that for $0 \leqslant j \leqslant k$,

$$\| D^j(f-s)\|_{L_\infty[a,b]}$$
$$\leqslant \| D^j(f-t)\|_{L_\infty[a,b]} + \| D^j h \|_{L_\infty[a,b]} + \| D^j(h-(t-s))\|_{L_\infty[a,b]} . \tag{7.11}$$

The first term on the right side of (7.11) can evidently be bounded above by (7.4) of Lemma 7.2. To bound the second term on the right side of (7.11), we note from a simple calculation based on the definitions of $\tilde{\phi}_{i,j}(x)$ that

$$\| D^l \tilde{\phi}_{i,j} \|_{L_q[a,b]} \leqslant K\pi^{j-l+(1/q)}, \qquad 1 \leqslant q \leqslant \infty, \quad 0 \leqslant l \leqslant 2m. \tag{7.12}$$

Next, note that from the definition of s and t that

$$| D^j(s-t)(a)| = | D^j(f-t)(a)| \leqslant K\pi^{k-j}\omega(D^k f, \pi), \ 0 \leqslant j \leqslant \min(k, m-1), \tag{7.13}$$

the last inequality following from (7.4) of Lemma 7.2. Similarly,

$$| D^j(s-t)(a)| = | D^j t(a)| \leqslant K\pi^{k-j}\omega(D^k f, \pi), \text{ if } \min(k, m-1) < j \leqslant 2m-1, \tag{7.14}$$

with analogous inequalities holding at the endpoint b. Thus, from the combined inequalities of (7.12)–(7.14), it follows then from the definition of h in (7.10) that

$$\begin{aligned} \| D^j h \|_{L_\infty[a,b]} &\leqslant K\pi^{k-j}\omega(D^k f, \pi), \qquad && 0 \leqslant j \leqslant 2m, \\ \| D^j h \|_{L_2[a,b]} &\leqslant K\pi^{k-j+(1/2)}\omega(D^k f, \pi), && 0 \leqslant j \leqslant 2m. \end{aligned} \tag{7.15}$$

The first inequality of (7.15) then bounds the second term on the right side of (7.11). For the third term of (7.11), we note that $t - s$ is in fact the $\mathrm{Sp}^{(m)}(\Delta_u)$-interpolant of h in the sense of (3.1). As such, the error bounds of (3.2) of Lemma 3.1 are applicable, i.e.,

$$\| D^j(h - (t - s))\|_{L_\infty[a,b]} \leqslant K\pi^{2m-j-1/2} \| D^{2m}h \|_{L_2[a,b]}, \qquad 0 \leqslant j \leqslant 2m - 1.$$

But, from the second inequality of (7.15), we deduce that

$$\| D^{2m}h \|_{L_2[a,b]} \leqslant K\pi^{k-2m+1/2}\omega(D^kf, \pi),$$

and consequently,

$$\| D^j(h - (t - s))\|_{L_\infty[a,b]} \leqslant K\pi^{k-j}\omega(D^kf, \pi), \qquad 0 \leqslant j \leqslant 2m - 1.$$

Combining these bounds gives the desired result of the first inequality of (7.8). The remaining inequalities of (7.8) are similarly established.

Finally, suppose that $\tilde{s} \in \mathrm{Sp}^{(m)}(\Delta_u)$ is defined, as described in the statement of Theorem 7.4, by means of Lagrange polynomial interpolation of f at the boundaries of $[a, b]$. In this case, we define $\tilde{h} \in H^{(2m+1)}(\Delta_u)$, in analogy with (7.10), by

$$\tilde{h}(x) = \sum_{j=0}^{2m-1} \{D^j(t - \tilde{s})(a) \cdot \tilde{\phi}_{0,j}(x) + D^j(t - \tilde{s})(b) \cdot \tilde{\phi}_{1,j}(x)\}. \tag{7.16}$$

Since $| D^j(\tilde{s} - t)(a)| \leqslant | D^j(\tilde{s} - f)(a)| + | D^j(f - t)(a)|$, and

$$| D^j(\tilde{s} - t)(a)| \leqslant | D^j\tilde{s}(a)| + | D^jt(a)|,$$

it follows directly from Theorem 4.1 and Lemma 7.2 that

$$| D^j(\tilde{s} - t)(a)| \leqslant K\pi^{k-j}\, \omega(D^kf, \pi), \qquad 0 \leqslant j \leqslant 2m - 1.$$

But since these *same* bounds were used in (7.13)–(7.14) to establish the bounds of (7.8) for s, it follows that the bounds of (7.8) remain unchanged if s is replaced by $\tilde{s}$. Q.E.D.

COROLLARY 7.5. *With the hypotheses of Theorem* 7.4, *if* $f \in W_r^{k+1}[a, b]$ *with* $0 \leqslant k < 2m$ *and* $1 \leqslant r \leqslant \infty$, *then for* $\max(r, 2) \leqslant q \leqslant \infty$ *and for either* s *or* $\tilde{s}$,

$$K\pi^{k+1-j-(1/r)+(1/q)} \| D^{k+1}f\|_{L_r[a,b]} \geqslant \begin{cases} \| D^j(f - s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^js \|_{L_q[a,b]}, & \text{if } k < j \leqslant 2m - 1. \end{cases} \tag{7.17}$$

It is worth mentioning the relationship of Theorem 7.4 with the many known results in the literature on interpolation and approximation errors for polynomial spline functions. Most authors have considered error bounds in the uniform norm. For cubic splines ($m = 2$), the validity of the case $k = 2$ of Theorem 7.4 was first indicated in Ahlberg and Nilson [1, p. 104], while the case $k = 3$ was given by Birkhoff and deBoor [4, 5], both results being for more general partitions Δ of $[a, b]$ than those considered in Theorem 7.4; see also Swartz [38, Section 10]. For periodic cubic splines, the earlier results ($k = 2$) of Walsh, Ahlberg, and Nilson [41] were extended by Sharma and Meir [34], Nord [24] (with his well-known counterexample), Cheney and Schurer [12], Meir and Sharma [23], and Ahlberg, Nilson, and Walsh [3]. For cubic splines defined by interpolation not necessarily at knots, Meir and Sharma [22] have related results.

Concerning the case of higher-order spline interpolation of odd degree for general partitions, Schoenberg [27] and Ahlberg, Nilson, and Walsh [2, 3] have presented results related to Theorem 7.4 and Corollary 7.5, generally under different boundary conditions.

For higher-order odd-degree splines and uniformly spaced knots, Ahlberg, Nilson, and Walsh [3] gave a result like that of Lemma 7.1 for periodic boundary conditions, while Subbotin [36], working with even-degree interpolating splines with interpolation taking place half-way between the knots, proved related results. For general odd- as well as even-degree interpolating splines, Golomb [14, 15] has error bounds in the periodic case which correspond exactly to those of Theorem 7.4 and Corollary 7.5, but the proofs are of a different nature. Golomb [14, 15] also presented the first few terms in the asymptotic expansions of the errors (cf. also Swartz [37]), and he also exhibited [15, Theorem 4] cases in which the use of periodic boundary conditions for interpolating splines (of degree at least five) yields dramatically poorer convergence rates than Hermite boundary conditions.

Approximating splines which are defined by linear processes other than interpolation have been considered by various authors. Schoenberg [28] showed the convergence of his high-order splines (with uniformly placed knots) to a continuous function. In the fundamental work of deBoor [9], error estimates like those of Theorem 7.4 have been obtained for even- as well as odd-degree splines with *no* assumptions on the partitions. The splines used there are obtained by linear projections, these projections being explicitly given for linear and quadratic splines (cf. deBoor [8, 9]). We note that, in contrast with interpolatory splines, deBoor's approximations converge locally, at rates depending on the local smoothness of the function approximated. Schultz [30, p. 198] gives error estimates for integral least-square spline approximations. Finally, we mention that other linear maps onto the splines have been successfully used by Strang and Fix [35] for periodic spline approximation in higher dimensions.

To conclude this section, we state a stability result which is established in the manner of Theorems 5.1 and 7.4.

THEOREM 7.6. *Given $f \in C^k[a, b]$ with $0 \leqslant k < 2m$ and given $\Delta_u \in \mathscr{P}_1(a, b)$, let s be the unique element in $\mathrm{Sp}^{(m)}(\Delta_u)$ such that*

$$\begin{aligned} s(x_i) &= \alpha_{i,0}, \qquad 0 \leqslant i \leqslant N \\ D^j s(a) = \alpha_{0,j}, &\qquad D^j s(b) = \alpha_{N,j}, \qquad \text{if} \quad 1 \leqslant j \leqslant m-1, \end{aligned} \tag{7.18}$$

where it is assumed that functions $F_i(f, \pi)$ exist such that

$$\begin{aligned} K\pi^k F_i(f, \pi) &\geqslant | f(x_i) - \alpha_{i,0} |, \qquad 0 \leqslant i \leqslant N, \\ K\pi^{k-j} F_0(f, \pi) &\geqslant | D^j f(a) - \alpha_{0,j} |, \quad \text{if} \quad 1 \leqslant j \leqslant \min(k, m-1), \\ K\pi^{k-j} F_0(f, \pi) &\geqslant | \alpha_{0,j} |, \quad \text{if} \quad \min(k, m-1) < j \leqslant m-1, \end{aligned} \tag{7.19}$$

with similar inequalities holding at $x = b$. Then, with

$$\| F \|_\infty = \max_{0 \leqslant i \leqslant N} F_i(f, \pi),$$

$$K\pi^{k-j}(\omega(D^k f, \pi) + \| F \|_\infty) \geqslant \begin{cases} \| D^j(f - s) \|_{L_\infty[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_\infty[a,b]}, & \text{if} \quad k < j \leqslant 2m-1. \end{cases} \tag{7.20}$$

In particular, if Δ_u has at least $2m$ knots, and if s and its first $(m-1)$ derivatives at $x = a$ are defined by (7.9) *in terms of Lagrange polynomial interpolation of $\{\alpha_{i,0}\}_{i=0}^{2m-1}$, where the $\alpha_{i,0}$ satisfy the first inequality of* (7.19), *with a similar definition holding at $x = b$, then the bounds of* (7.20) *are valid.*

COROLLARY 7.7. *With the hypotheses of Theorem* 7.6, *if $f \in W_r^{k+1}[a, b]$ with $1 \leqslant r \leqslant \infty$ and $0 \leqslant k < 2m$, and if functions $F_i(f, \pi)$ exist* [cf. (7.19)] *such that for $0 \leqslant i \leqslant N$,*

$$\begin{aligned} K\pi^{k+1} F_i(f, \pi) &\geqslant | f(x_i) - \alpha_{i,0} |, \qquad 0 \leqslant i \leqslant N, \\ K\pi^{k+1-j} F_0(f, \pi) &\geqslant | D^j f(a) - \alpha_{0,j} |, \quad \text{if} \quad 1 \leqslant j \leqslant \min(k, m-1), \\ K\pi^{k+1-j} F_0(f, \pi) &\geqslant | \alpha_{0,j} |, \quad \text{if} \quad \min(k, m-1) < j \leqslant m-1, \end{aligned} \tag{7.21}$$

with similar inequalities holding at $x = b$, then for $\max(r, 2) \leqslant q \leqslant \infty$,

$$K\pi^{k+1-j-(1/r)+(1/q)}\{\| D^{k+1} f \|_{L_r[a,b]} + \| F \|_r\}$$

$$\geqslant \begin{cases} \| D^j(f - s) \|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_q[a,b]}, & \text{if} \quad k < j \leqslant 2m-1. \end{cases} \tag{7.22}$$

As an application of Theorem 7.6, consider the case of cubic splines ($m = 2$) on a uniform partition, $\mathrm{Sp}^{(2)}(\Delta_u)$, where Δ_u has at least four knots, i.e., $N \geqslant 3$. Then, given $f \in C^k[a, b]$, $0 \leqslant k < 4$, let $s \in \mathrm{Sp}^{(2)}(\Delta_u)$ be the unique element such that

$$\begin{aligned} s(x_i) &= f(x_i), \qquad \equiv \alpha_{i,0}, \; 0 \leqslant i \leqslant N, \\ Ds(a) &= (1/6\pi)\{-11f(x_0) + 18f(x_1) - 9f(x_2) + 2f(x_3)\} \equiv \alpha_{0,1}, \\ Ds(b) &= (1/6\pi)\{11f(x_N) - 18f(x_{N-1}) + 9f(x_{N-2}) - 2f(x_{N-3})\} \equiv \alpha_{N,1}, \end{aligned}$$

where $\pi = (x_{i+1} - x_i)$, $0 \leqslant i \leqslant N - 1$. Because the right sides of the last two expressions above are simply obtained by differentiating the cubic Lagrange interpolation of f in the four knots x_0, x_1, x_2, x_3 and $x_{N-3}, x_{N-2}, x_{N-1}, x_N$, and evaluating the result, respectively, at $x = x_0$ and $x = x_N$, then the inequalities of (4.2) of Theorem 4.1 are applicable with $s = k$, from which the inequalities of (7.19) follow with $F_i(f, \pi) = \omega(D^k f, \pi)$. Consequently [cf. (7.20)],

$$K\pi^{k-j}\omega(D^k f, \pi) \geqslant \begin{cases} \| D^j(f - s)\|_{L_\infty[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_\infty[a,b]}, & \text{if} \quad k < j \leqslant 3, \end{cases}$$

which was stated in Section 1.

We remark that the special case of the stability result of Theorem 7.6 for cubic spline interpolation ($m = 2$) has also been independently developed by matrix techniques in Carlson and Hall [11].

8. Polynomial Spline Interpolation Errors under Second Integral Relation Boundary Conditions

In this final section, we discuss extensions of our previous results for polynomial splines ($L = D^m$) to more general boundary conditions.

Consider now any set of $2m$ real point functionals $B = \{B_j\}_{j=0}^{2m-1}$ on $W_2^{2m}[a, b]$, called *boundary conditions*, of the form

$$B_j g = \sum_{i=0}^{2m-1} \{a_{j,i}D^i g(a) + b_{j,i}D^i g(b)\}, \qquad 0 \leqslant j \leqslant 2m - 1, \quad g \in W_2^{2m}[a, b]. \tag{8.1}$$

If the $2m \times 4m$ matrix M is defined by

$$M = \begin{bmatrix} a_{0,0} & b_{0,0} & a_{0,1} & b_{0,1} & \cdots & b_{0,2m-1} \\ a_{1,0} & & & & & \\ \vdots & & & & & \vdots \\ a_{2m-1,0} & & & & \cdots & b_{2m-1,2m-1} \end{bmatrix}, \tag{8.2}$$

we assume that

$$\operatorname{rank} M = 2m, \tag{8.3}$$

i.e., the functionals of (8.1) are linearly independent. In addition, we assume that, for any $g \in W_2^{2m}[a, b]$ with $B_j g = 0$ for all $0 \leqslant j \leqslant 2m - 1$,

$$\int_a^b (D^m g)^2 \, dt = (-1)^m \int_a^b g \cdot D^{2m} g \, dt. \tag{8.4}$$

In other words, since integration by parts gives

$$\int_a^b (D^m g)^2 \, dt = (-1)^m \int_a^b g \cdot D^{2m} g \, dt + P(g)\Big|_{x=a}^{x=b},$$

where the bilinear concomitant $P(g)$ is defined by

$$P(g) = \sum_{j=0}^{m-1} (-1)^j \, D^{m-1-j} g(x) \cdot D^{m+j} g(x),$$

then (8.4) is equivalent to $P(g) = 0$ for all g such that

$$B_j g = 0, 0 \leqslant j \leqslant 2m - 1.$$

By means of elementary row operations applied to M (cf. Birkhoff and MacLane [6, p. 162]), there is no loss of generality in assuming that M is in *lower reduced echelon form*, i.e.,

(i) every leading entry (from the right) of each row is unity;

(ii) every column containing a leading entry (from the right) has all other entries zero; (8.5)

(iii) if the leading entry (from the right) of row i is in column t_i, then $t_1 < t_2 < \cdots < t_m$.

We remark that the elementary row operations which bring M into lower reduced echelon form leave the property of (8.4) invariant.

We now cite a few examples of boundary conditions $B = \{B_j\}_{j=0}^{2m-1}$ which satisfy (8.3)–(8.5).

EXAMPLE 1. Hermite boundary conditions:

$$B_{2j} g = D^j g(a), \qquad B_{2j+1} g = D^j g(b), \qquad 0 \leqslant j \leqslant m - 1.$$

For the case $m = 2$, M is given explicitly by

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

EXAMPLE 2. Odd-derivative boundary conditions:

$$B_0 g = g(a); \qquad B_1 g = g(b);$$
$$B_{2j} g = D^{2j-1} g(a); \qquad B_{2j+1} g = D^{2j-1} g(b),\ 1 \leqslant j \leqslant m - 1.$$

EXAMPLE 3. Even-derivative boundary conditions:

$$B_{2j} g = D^{2j} g(a); \qquad B_{2j+1} g = D^{2j} g(b), \qquad 0 \leqslant j \leqslant m - 1.$$

EXAMPLE 4. Periodic-type boundary conditions I:

$$B_0 g = g(a); \qquad B_1 g = g(b);$$
$$B_j g = D^{j-1} g(a) - D^{j-1} g(b), \qquad 2 \leqslant j \leqslant 2m - 1.$$

EXAMPLE 5. Periodic-type boundary conditions II:

$$B_j g = D^j g(a) - D^j g(b), \qquad 0 \leqslant j \leqslant 2m - 1.$$

EXAMPLE 6. Natural boundary conditions:

$$B_0 g = g(a), \qquad B_{2j} g = D^{m+j-1} g(a), \qquad 1 \leqslant j \leqslant m - 1,$$
$$B_1 g = g(b), \qquad B_{2j+1} g = D^{m+j-1} g(b), \qquad 1 \leqslant j \leqslant m - 1.$$

We next define a way in which functions $f \in C^k[a, b]$ can be interpolated in $\mathrm{Sp}(D^m, \Delta, z)$ when $0 \leqslant k < 2m$. The approach is like that of the interpolation in (3.3) of Lemma 3.2, i.e., high "derivatives" of f which are needed to define the interpolation are treated as though they were zero.

LEMMA 8.1. *Given $f \in C^k[a, b]$ with $0 \leqslant k < 2m$, given $\Delta \in \mathscr{P}(a, b)$ with $N > m$* [cf. (2.1)], *and given point functionals $\{B_j\}_{j=0}^{2m-1}$ of the form* (8.1) *satisfying* (8.3)–(8.5), *then there exists a unique $s \in \mathrm{Sp}(D^m, \Delta, z)$ which interpolates f in the sense that*

$$\begin{aligned} D^j(f - s)(x_i) &= 0, \qquad 0 \leqslant j \leqslant \min(k, z_i - 1), \quad 1 \leqslant i \leqslant N - 1, \\ D^j s(x_i) &= 0, \quad \text{if} \quad \min(k, z_i - 1) < j \leqslant z_i - 1, \quad 1 \leqslant i \leqslant N - 1, \end{aligned} \tag{8.6}$$

and

$$B_j s = \sum_{i=0}^{k} \{a_{j,i} D^i f(a) + b_{j,i} D^i f(b)\}, \qquad 0 \leqslant j \leqslant 2m-1. \tag{8.7}$$

Proof. Since the complete proof follows rather familiar lines (cf. Schultz and Varga [32, Theorem 3]), we merely sketch it. The determination of such an $s \in \mathrm{Sp}(D^m, \Delta, z)$ depends on the solution of $2mN$ linear equations in $2mN$ unknowns, and as such, it is sufficient to show that if $f(x) \equiv 0$ on $[a, b]$, then $s(x) \equiv 0$ also on $[a, b]$. Consider

$$\int_a^b (D^m s(x))^2 \, dx = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} (D^m s(x))^2 \, dx.$$

Integrating by parts, and using the continuity requirements of (2.4), the interpolation requirements of (8.6), and the assumption of (8.4), gives us that $s(x)$ is a polynomial of degree at most $m - 1$, But from (8.6), since s interpolates $f \equiv 0$ in $(N - 1)$ distinct points then, as $N - 1 \geqslant m$, $s(x) \equiv 0$ on $[a, b]$. Q.E.D.

The proof of the next lemma, the *second integral relation*, also follows along the same familiar lines (cf. [32, Theorem 5]), and is omitted.

LEMMA 8.2. *Given* $f \in W_2^{2m}[a, b]$, *given* $\Delta \in \mathscr{P}(a, b)$ *with* $N > m$, *and given point functionals* $\{B_j\}_{j=0}^{2m-1}$ *of the form* (8.1) *satisfying* (8.3)–(8.5), *let* $s \in \mathrm{Sp}(D^m, \Delta, z)$ *be the unique interpolation of* f *in the sense of* (8.6)–(8.7) *with* $k = 2m - 1$. *Then,*

$$\int_a^b (D^m(f - s))^2 \, dx = (-1)^m \int_a^b (f - s) \, D^{2m} f \, dx. \tag{8.8}$$

We next state a result on error bounds for the interpolation of Lemmas 8.1–8.2, which is the analogue of Lemma 3.1. This too can be proved in analogy with the results of Ahlberg, Nilson, and Walsh [3, Chapter 5] and Schultz and Varga [32]. We remark that our proof of Lemma 8.3 depends on an inequality like that of Hille, Szegö, and Tamarkin [17], relating the norm of the derivative of a polynomial to the norm of the polynomial. For details, see Swartz [39].

LEMMA 8.3. *Given* $f \in W_2^{2m}[a, b]$, *given* $\Delta \in \mathscr{P}_\sigma(a, b)$ *with* $N > m$, *and given point functionals* $\{B_j\}_{j=0}^{2m-1}$ *of the form* (8.1) *satisfying* (8.3)–(8.5), *let*

$s \in \mathrm{Sp}(D^m, \Delta, z)$ *be the unique interpolation of* f *in the sense of* (8.6)–(8.7) *with* $k = 2m - 1$. *Then, for* $2 \leqslant q \leqslant \infty$,

$$\| D^j(f - s)\|_{L_q[a,b]} \leqslant K\pi^{2m-j-(1/2)+(1/q)} \| D^{2m}f\|_{L_2[a,b]}, \qquad 0 \leqslant j \leqslant 2m - 1. \tag{8.9}$$

We now extend the result of Lemma 8.3 to less smooth functions, just as Theorem 3.5 extends the result of Lemma 3.1.

THEOREM 8.4. *Given* $f \in C^k[a, b]$ *with* $0 \leqslant k < 2m$, *given* $\Delta \in \mathscr{P}_\sigma(a, b)$ *with* $N > m$, *and given point functionals* $\{B_j\}_{j=0}^{2m-1}$ *of the form* (8.1) *satisfying* (8.3)–(8.5), *let* $s \in \mathrm{Sp}(D^m, \Delta, z)$ *be the unique interpolation of* f *in the sense of* (8.6)–(8.7). *Then, for* $2 \leqslant q \leqslant \infty$,

$$K\pi^{k-j-(1/2)+(1/q)}\omega(D^kf, \pi) \geqslant \begin{cases} \| D^j(f - s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_q[a,b]}, & \text{if } \; k < j \leqslant 2m - 1. \end{cases} \tag{8.10}$$

Proof. Given $f \in C^k[a, b]$, let $g \in C^{2m}[a, b]$ be its interpolation in the sense of (3.3) of Lemma 3.2. The triangle inequality gives us for $2 \leqslant q \leqslant \infty$ that

$$\| D^j(f - s)\|_{L_q[a,b]} \leqslant \| D^j(f - g)\|_{L_q[a,b]} + \| D^j(g - s)\|_{L_q[a,b]}, \qquad 0 \leqslant j \leqslant k, \tag{8.11}$$

where $s \in \mathrm{Sp}(D^m, \Delta, z)$ is the interpolation of f in the sense of (8.6)–(8.7). The first term on the right-hand side of the above inequality can be bounded above by (3.4) of Lemma 3.2. Next, note that s, while interpolating f in the sense of (8.6)–(8.7), necessarily also interpolates g in the sense of Lemma 8.3. Thus, from (8.9) of Lemma 8.3,

$$\| D^j(g - s)\|_{L_q[a,b]} \leqslant K\pi^{2m-j-(1/2)+(1/q)} \| D^{2m}g \|_{L_2[a,b]}, \qquad 0 \leqslant j \leqslant 2m - 1.$$

But from the case $j = 2m$ of (3.4) of Lemma 3.2, we have that

$$\| D^{2m}g \|_{L_2[a,b]} \leqslant K\pi^{k-2m}\omega(D^kf, \pi),$$

and thus

$$\| D^j(g - s)\|_{L_q[a,b]} \leqslant K\pi^{k-j-(1/2)+(1/q)}\omega(D^kf, \pi),$$

which yields the first inequality of (8.10). The remainder of the proof is similar. Q.E.D.

The following extension of Theorem 8.3 is easily established.

COROLLARY 8.5. *With the hypotheses of Theorem* 8.3, *if* $f \in W_r^{k+1}[a, b]$ *with* $1 \leqslant r \leqslant \infty$ *and* $0 \leqslant k < 2m$, *then for* $\max(r, 2) \leqslant q \leqslant \infty$,

$$K\pi^{k+1-j+(1/q)+\min(-1/r,-1/2)} \| D^{k+1}f \|_{L_r[a,b]} \geqslant \begin{cases} \| D^j(f - s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_q[a,b]}, & \text{if } k < j \leqslant 2m - 1. \end{cases} \tag{8.12}$$

Actually, a more general form of Theorem 8.4 can be proved, along the lines of Corollary 3.4. This is

COROLLARY 8.6. *Given* $f(x)$ *defined on* $[a, b]$ *such that* $D^{k-1}f$ (*for* $k \geqslant 1$) *is absolutely continuous and* D^kf, $0 \leqslant k < 2m$, *is defined and piecewise-continuous on* $[a, b]$ *with* $M \geqslant 0$ *discontinuities in the points* $\{y_j\}_{j=1}^M \subset (a, b)$, *assume that* D^kf *is absolutely continuous on each subinterval* (y_i, y_{i+1}), $0 \leqslant i \leqslant M$ (*where* $y_0 = a$, $y_{M+1} = b$), *with* $D^{k+1}f \in L_r[y_i, y_{i+1}]$, $0 \leqslant i \leqslant M$, *where* $1 \leqslant r \leqslant \infty$. *If* $\Delta \in \mathscr{P}_\sigma(a, b)$ *with* $N > m$, *and if the point functionals* $\{B_j\}_{j=0}^{2m-1}$ *of* (8.1) *satisfy* (8.3)–(8.5), *let* $s \in \mathrm{Sp}(D^m, \Delta, z)$ *be the unique interpolation of* f *in the sense of* (8.6)–(8.7). *Then, for* $\max(r, 2) \leqslant q \leqslant \infty$,

$$K\pi^{k-j+1/q}\{M^{1/q}\omega(D^kf, \pi) + \pi^{1-1/r} \| D^{k+1}f \|_{L_r[a,b]}\} \geqslant \begin{cases} \| D^j(f - s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_q[a,b]}, & \text{if } k < j \leqslant 2m - 1. \end{cases} \tag{8.13}$$

Proof. Following the proof of Theorem 8.4, let $g \in C^{2m}[a, b]$ be the interpolation of f in the sense of (3.3), and consider the inequality of (8.11). For $r \leqslant q \leqslant \infty$, the first term on the right-hand side of (8.11) can be bounded above by (3.15) of Corollary 3.4. To bound the second term on the right-hand side of (8.11), we use (8.9) of Lemma 8.3, coupled with the second inequality of (3.15), i.e., for $\max(r, 2) \leqslant q \leqslant \infty$ *and* $0 \leqslant j \leqslant k$,

$$\| D^j(g - s)\|_{L_q[a,b]} \leqslant K\pi^{k-j+1/q}\{M^{1/q}\omega(D^kf, \pi) + \pi^{1-1/r} \| D^{k+1}f \|_{L_r[a,b]}\},$$

which is the first inequality of (8.13). The remainder of the proof is similar. Q.E.D.

We remark that for the special case of periodic boundary conditions, as given in Example 4, the *global* error estimates of (8.13) of Corollary 8.6 contain, in the case of uniform partitions, certain error bounds obtained by Golomb [15]. We should add, however, that Golomb [15] obtains in the cases he considers a more precise description of the *local* behavior of the interpolation errors at certain points.

We now turn to stability results related to Theorem 8.4. To establish such stability results, we first recall (cf. Section 7) the polynomials $\varphi_{i,k}(x)$ of degree $(4m+1)$ defined by

$$D^l\varphi_{i,k}(0) = \delta_{l,k}\cdot\delta_{i,0}\,,\ D^l\varphi_{i,k}(1) = \delta_{l,k}\delta_{i,1}\,,\ 0 \leqslant i \leqslant 1,\ 0 \leqslant l, k \leqslant 2m,$$

and we set

$$\tilde{\varphi}_{0,k}(x) = \begin{cases} h_0{}^k\varphi_{0,k}((x-a)/h_0), & x \in [a, a+h_0 = x_1], \\ 0, & \text{otherwise}, \end{cases}$$

$$\tilde{\varphi}_{1,k}(x) = \begin{cases} h_{N-1}^k\varphi_{1,k}((x-b+h_{N-1})/h_{N-1}), & x \in [b-h_{N-1} = x_{N-1}\,, b], \\ 0, & \text{otherwise}. \end{cases}$$

Similar piecewise polynomials where used in Section 7 for uniform partitions of $[a, b]$. Note that all $\tilde{\varphi}_{i,k}(x)$ are identically zero in $[x_1\,, x_{N-1}]$, and are of class $C^{2m}[a, b]$. Next, applying the functional B_j of (8.1) to $\tilde{\varphi}_{i,k}(x)$ gives

$$B_j\tilde{\varphi}_{0,k} = a_{j,k}; \qquad B_j\tilde{\varphi}_{1,k} = b_{j,k}\,, \qquad 0 \leqslant j,\quad k \leqslant 2m-1. \tag{8.14}$$

Note that the value of these functionals in (8.14) is independent of the partition. Thus, if we consider any

$$\psi(x) = \sum_{k=0}^{2m-1}\{\alpha_k\tilde{\varphi}_{0,k}(x) + \beta_k\tilde{\varphi}_{1,k}(x)\},$$

which is in $C^{2m}[a, b]$ and identically zero in $[x_1\,, x_{N-1}]$, then finding a $\psi_l(x)$ with $0 \leqslant l \leqslant 2m-1$ such that

$$B_j\psi_l = \delta_{j,l}\,, \qquad 0 \leqslant j \leqslant 2m-1, \tag{8.15}$$

amounts to solving the matrix problem [cf. (8.2)]

$$M(\alpha_0^{(l)}, \beta_0^{(l)}, \alpha_1^{(l)}, \beta_1^{(l)},\ldots, \beta_{2m-1}^{(l)})^T = (\delta_{0,l}\,,\ldots, \delta_{2m-1,l})^T. \tag{8.16}$$

This, by virtue of the assumption (8.3) that rank $M = 2m$, has a nonzero solution for each l with $0 \leqslant l \leqslant 2m-1$. In fact, making use of the assumption (8.5-iii) that M is in lower reduced echelon form, we can find a solution $\psi_l(x)$ of (8.15) which can be expressed as

$$\psi_l(x) = \sum_{k=0}^{\tau(l)}\{\alpha_k^{(l)}\tilde{\varphi}_{0,k}(x) + \beta_k^{(l)}\tilde{\varphi}_{1,k}(x)\}, \qquad 0 \leqslant l \leqslant 2m-1, \tag{8.17}$$

where $\tau(l)$, the *order* of B_l, is simply determined from the l-th row of M by means of

$$\tau(l) = \min\{k : |\,a_{l,r}\,| + |\,b_{l,r}\,| = 0 \qquad \text{for all} \quad r > k\}. \tag{8.18}$$

This of course means that we can also write [cf. (8.1)]

$$B_j g = \sum_{i=0}^{\tau(j)} \{a_{j,i} D^i g(a) + b_{j,i} D^i g(b)\}, \qquad 0 \leqslant j \leqslant 2m-1. \qquad (8.1')$$

This order $\tau(l)$ of B_l is used in

LEMMA 8.7. *Given the point functionals* $\{B_j\}_{j=0}^{2m-1}$ *of the form* (8.1) *satisfying* (8.3)–(8.5), *let* $\psi_l(x)$ *be given by* (8.17), *so that* ψ_l *satisfies* (8.15). *Then, for* $\Delta \in \mathscr{P}_\sigma(a, b)$ *and for* $1 \leqslant q \leqslant \infty$,

$$\| D^k \psi_l \|_{L_q[a,b]} \leqslant K \pi^{\tau(l)-k+1/q}, \qquad 0 \leqslant k \leqslant 2m, \quad 0 \leqslant l \leqslant 2m-1. \qquad (8.19)$$

Proof. Using the representation of (8.17), the result of (8.19) follows immediately from the bounds of (7.12) for the $\tilde{\varphi}_{i,k}(x)$ and the definition of $\tau(l)$ in (8.18) Q.E.D.

This brings us now to the following stability form of Theorem 8.4, in which the boundary conditions $\{B_j\}_{j=0}^{2m-1}$ are only approximately satisfied. This extends the result of Theorem 5.1 to polynomial splines satisfying more general boundary conditions.

THEOREM 8.8. *Given* $f \in C^k[a, b]$ *with* $0 \leqslant k < 2m$, *given* $\Delta \in \mathscr{P}_\sigma(a, b)$ *with* $N > m$, *and given the point functionals* $\{B_j\}_{j=0}^{2m-1}$ *of the form* (8.1) *satisfying* (8.3)–(8.5), *let* $s \in \mathrm{Sp}(D^m, \Delta, z)$ *be the unique interpolation of* f *in the following sense:*

$$D^j s(x_i) = \alpha_{i,j}, \qquad 0 \leqslant j \leqslant z_i - 1, \quad 1 \leqslant i \leqslant N-1, \qquad (8.20)$$

where it is assumed that functions $F_i(f, \pi)$ *exist such that for* $1 \leqslant i \leqslant N-1$,

$$K\pi^{k-j} F_i(f, \pi) \geqslant \begin{cases} | D^j f(x_i) - \alpha_{i,j} |, & 0 \leqslant j \leqslant \min(k, z_i - 1), \\ | \alpha_{i,j} |, & \text{if } \min(k, z_i - 1) < j \leqslant z_i - 1, \end{cases} \qquad (8.21)$$

while at the boundary,

$$B_j s = \beta_j, \qquad 0 \leqslant j \leqslant 2m-1, \qquad (8.22)$$

where it is assumed [cf. (8.18)] *that, with* $\tau(j)$ *the order of* B_j,

$$K\pi^{k-\tau(j)} F_0(f, \pi) \geqslant \left| \beta_j - \sum_{i=0}^{k} \{a_{j,i} D^i f(a) + b_{j,i} D^i f(b)\} \right|,$$

$$0 \leqslant j \leqslant 2m-1. \qquad (8.23)$$

Then, with $\| F \|_2^2 = \pi \sum_{i=0}^{N-1} F_i^2(f, \pi)$ *and for* $2 \leqslant q \leqslant \infty$,

$$K\pi^{k-j-(1/2)+(1/q)}\{\omega(D^k f, \pi) + \| F \|_2\}$$
$$\geqslant \begin{cases} \| D^j(f-s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k \\ \| D^j s \|_{L_q[a,b]}, & \textit{if} \quad k < j \leqslant 2m-1. \end{cases} \tag{8.24}$$

Proof. The proof is like that of Theorems 3.5 and 5.1, and is merely sketched. Let $t \in \mathrm{Sp}(D^m, \Delta, z)$ be the interpolation of f in the sense of (8.6)–(8.7) of Lemma 8.1. Let $h_0(x) \in H^{(2m+1)}(\Delta)$ be the polynomial spline such that for $1 \leqslant i \leqslant N-1$ [cf. (5.4)]

$$D^j h_0(x_i) = \begin{cases} D^j f(x_i) - \alpha_{i,j}, & 0 \leqslant j \leqslant \min(k, z_i - 1), \\ -\alpha_{i,j}, & \text{if} \quad \min(k, z_i - 1) < j \leqslant z_i - 1, \\ 0, & z_i - 1 < j \leqslant 2m, \end{cases}$$

and

$$D^j h_0(a) = D^j h_0(b) = 0, \qquad 0 \leqslant j \leqslant 2m.$$

Let

$$h(x) = h_0(x) + \sum_{j=0}^{2m-1} \left[\sum_{l=0}^{k} \{a_{j,l} D^l f(a) + b_{j,l} D^l f(b) - \beta_j\} \right] \psi_j(x).$$

Then upon writing $f - s = (f - t) + h + [(t - s) - h]$, we have for $0 \leqslant j \leqslant k$ and $2 \leqslant q \leqslant \infty$ that

$$\| D^j(f-s)\|_{L_q[a,b]}$$
$$\leqslant \| D^j(f-t)\|_{L_q[a,b]} + \| D^j h \|_{L_q[a,b]} + \| D^j\{h - (t-s)\}\|_{L_q[a,b]}. \tag{8.25}$$

The first term on the right side of (8.25) can be suitably bounded above by (8.10) of Theorem 8.4. The second term on the right side of (8.25) can be bounded above as in the proof of Theorem 5.1, using the bounds on the $\psi_j(x)$'s (8.19) of Lemma 8.7, coupled with the assumption of (8.23), for $x \in [x_0, x_1]$ or $x \in [x_{N-1}, x_N]$. In the same fashion, Lemmas 8.3 and 8.7 aid in suitably bounding the third term on the right of (8.25). The remainder of the proof now exactly parallels the proof of Theorem 5.1. Q.E.D.

COROLLARY 8.9. *With the hypotheses of Theorem* 8.8, *if* $f \in W_r^{k+1}[a, b]$ *with* $1 \leqslant r \leqslant \infty$ *and* $0 \leqslant k < 2m$, *and if functions* $F_i(f, \pi)$ *exist* [cf. (8.21)–(8.23)] *such that for* $1 \leqslant i \leqslant N-1$,

$$K\pi^{k+1-j} F_i(f, \pi) \geqslant \begin{cases} | D^j f(x_i) - \alpha_{i,j} |, & 0 \leqslant j \leqslant \min(k, z_i - 1), \\ | \alpha_{i,j} |, & \textit{if} \quad \min(k, z_i - 1) < j \leqslant z_i - 1, \end{cases} \tag{8.26}$$

while at the boundary,

$$K\pi^{k+1-\tau(j)}F_0(f,\pi) \geqslant \left|\beta_j - \sum_{i=0}^{k}\{a_{j,i}D^if(a) + b_{j,i}D^if(b)\}\right|,$$
$$0 \leqslant j \leqslant 2m-1, \qquad (8.27)$$

then with $\| F \|_r^r \equiv \pi \sum_{i=0}^{N-1} F_i^r(f,\pi)$, *and for* $\max(r, 2) \leqslant q \leqslant \infty$,

$$K\pi^{k+1-j+(1/q)+\min(-1/r,-1/2)}\{\| D^{k+1}f\|_{L_r[a,b]} + \| F \|_r\}$$
$$\geqslant \begin{cases} \| D^j(f-s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^js \|_{L_q[a,b]}, & \text{if } \; k < j \leqslant 2m-1. \end{cases} \qquad (8.28)$$

The preceding results of this section for polynomial splines were proved for general partitions $\Delta \in \mathscr{P}_\sigma(a, b)$. If, however, we restrict attention to *uniform* partitions $\Delta_u \in \mathscr{P}_1(a, b)$, then the sharper results of Section 7 generalize to our "second integral relation" boundary conditions. The proofs are similar to those in Section 7, with Lemma 8.3 replacing Lemma 3.1, and the ψ_j of Lemma 8.7 entering into the analogues of (7.10) and (7.16). For details, see Swartz [39]. We state the analogue of Theorem 7.6 as

THEOREM 8.10. *Given* $f \in C^k[a, b]$ *with* $0 \leqslant k < 2m$, *given* $\Delta_u \in \mathscr{P}_1(a, b)$ *with* $N > m$, *and given the point functionals* $\{B_j\}_{j=0}^{2m-1}$ *of the form* (8.1) *satisfying* (8.3)–(8.5), *let* $s \in \mathrm{Sp}^{(m)}(\Delta_u)$ *be the unique interpolation of* f *in the following sense*:

$$\begin{aligned} s(x_i) &= \alpha_i, & 1 \leqslant i \leqslant N-1, \\ B_js &= \beta_j, & 0 \leqslant j \leqslant 2m-1, \end{aligned} \qquad (8.29)$$

where it is assumed that functions $F_i(f, \pi)$ *exist such that*

$$K\pi^kF_i(f,\pi) \geqslant |f(x_i) - \alpha_i|, \qquad 1 \leqslant i \leqslant N-1,$$
$$K\pi^{k-\tau(j)}F_0(f,\pi) \geqslant \left|\beta_j - \sum_{i=0}^{k}\{a_{j,i}D^if(a) + b_{j,i}D^if(b)\}\right|, \qquad (8.30)$$
$$0 \leqslant j \leqslant 2m-1,$$

where $\tau(j)$ *is the order of* B_j. *Then,*

$$K\pi^{k-j}(\omega(D^kf,\pi) + \| F \|_\infty) \geqslant \begin{cases} \| D^j(f-s)\|_{L_\infty[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^js \|_{L_\infty[a,b]}, & \text{if } \; k < j \leqslant 2m-1. \end{cases} \qquad (8.31)$$

In particular, if Δ_u *has at least* $(2m + 2)$ *knots, and if* $L_{2m-1,1}f\,(L_{2m-1,N-1}f)$

is the Lagrange polynomial interpolation of the data $\alpha_1, \ldots, \alpha_{2m}$ *in the knots* $x_1, x_2, \ldots, x_{2m}$ *(resp.,* $\alpha_{N-2m}, \ldots, \alpha_{N-1}$ *in the knots* $x_{n-2m}, x_{N-2m+1}, \ldots, x_{N-1}$*), and if we take*

$$B_j s = \beta_j \equiv \sum_{i=0}^{\tau(j)} \{a_{j,i} D^i(L_{2m-1,1} f)(a) + b_{j,i} D^i(L_{2m-1,N-1} f)(b)\},$$
$$0 \leqslant j \leqslant 2m-1, \quad (8.32)$$

then the error bounds of (8.31) *are valid, assuming only the first inequality of* (8.30).

For the last part of Theorem 8.10 involving the use of Lagrange polynomial interpolation of the data, it is necessary to remark that the quantities $D^j(L_{2m-1,1}f)(a)$ and $D^j(L_{2m-1,N-1}f)(b)$ in (8.32) involve *extrapolation* of the data at interior knots to boundary knots. [We have used extrapolations because neither $f(a)$ nor $f(b)$ may be involved in any $B_j f$, $0 \leqslant j \leqslant 2m-1$.] The proof of this portion of Theorem 8.10 than depends on the observation that the error bounds of Theorems 4.1 and 4.4 *remain valid* for these particular extrapolations.

COROLLARY 8.11. *With the hypotheses of Theorem* 8.10, *if* $f \in W_r^{k+1}[a, b]$ *with* $1 \leqslant r \leqslant \infty$ *and* $0 \leqslant k < 2m$, *and if functions* $F_i(f, \pi)$ *exist such that*

$$K\pi^{k+1} F_i(f, \pi) \geqslant |f(x_i) - \alpha_i|, \qquad 1 \leqslant i \leqslant N-1,$$
$$K\pi^{k+1-\tau(j)} F_0(f, \pi) \geqslant \left| \beta_j - \sum_{i=0}^{k} \{a_{j,i} D^i f(a) + b_{j,i} D^i f(b)\} \right|, \qquad (8.33)$$
$$0 \leqslant j \leqslant 2m-1,$$

then for $\max(r, 2) \leqslant q \leqslant \infty$,

$$K\pi^{k+1-j-(1/r)+(1/q)}\{\| D^{k+1} f \|_{L_r[a,b]} + \| F \|_r\}$$
$$\geqslant \begin{cases} \| D^j(f-s) \|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_q[a,b]}, & \text{if } k < j \leqslant 2m-1. \end{cases} \qquad (8.34)$$

We remark that even if the functionals of (8.1) are *not* expressed in echelon form (8.5), the results of Lemma 8.1 through Corollary 8.6 remain valid. The same may not be true for the associated stability results. The conclusion of Theorem 8.8 does hold true if the perturbations of the functionals B_j are due really to suitable perturbations in the derivatives which occur in each functional. Instead, if one *independently* perturbs each functional by a corresponding order of magnitude, the analogous result may not hold.

This phenomenon can be observed in the piecewise-linear interpolation ($m = 1$) on $\Delta \in \mathscr{P}(a, b)$, under boundary conditions associated with

$$B_0 g \equiv Dg(a); \qquad B_1 g \equiv Dg(a) + g(b).$$

We finally observe that the results of Theorem 8.10 and Corollary 8.11 hold for nonuniform partitions $\Delta \in \mathscr{P}_\sigma(a, b)$ in the cases of cubic ($m = 2$) and quintic ($m = 3$) splines. The proofs are those of Theorem 8.10 and Corollary 8.11, except that Lemma 7.1 is replaced with the analogous result for Hermite boundary conditions, found in Birkhoff and de Boor [4], Ahlberg, Nilson, and Walsh [3], and Swartz [38, Section 10] for $m = 2$, and in de Boor [10] for $m = 3$.

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